LINEAR ALGEBRA WORKSHEET 3

MATH1014 SPRING SESSION

(1) Let $T : \mathbb{R}^3 \to M_{2\times 2}$ be the function

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 & a-2b \\ a-2b & b-c \end{bmatrix}.$$

(a) T is a linear transformation. What would you need to check in order to verify this?

Solution. You would need to check that

$$T\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right)$$

and

$$T\left(d\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = dT\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right).$$

□

(b) Which, if any, of the following vectors are in Ker(T)?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$$

Solution. You can apply T to each of the three vectors and see when $T(\mathbf{x}) = \mathbf{0}$. It turns out that only $\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ is in Ker(T). □

(2) Suppose that we know the matrix A reduces to B

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \rightsquigarrow B = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for Nul(A).

Solution. The third and fifth variables are free. The reduced system of equations is $x_1 - 3x_3 + 19x_5 = 0$, $x_2 + 2x_3 - 3x_5 = 0$, and $x_4 = 0$. The general vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So a basis for Nul(A) is

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$\square$

(b) Find a basis for Col(A).

Solution. The first, second, and fourth columns are pivot columns. So

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 3 \\ -9 \end{bmatrix} \right\}$$

is a basis for Col(A) $\square$

(c) Find the dimensions of the spaces from part (a) and (b), and verify that the Rank Theorem holds.

Solution. We see from (a) and (b) that $\dim \mathrm{Col}(A) = \mathrm{Rank}(A) = 3$ and $\dim \mathrm{Nul}(A) = 2$. These add to 5, which is indeed the number of columns of A. $\square$

(3) Are the polynomials listed below linearly independent in $\mathbb{P}_2$? Do they span $\mathbb{P}_2$?

$$1-3t, \quad 1+t^2, \quad 1-3t+t^2$$

Solution. Suppose

$$a(1-3t)+b(1+t^2)+c(1-3t+t^2) = 0(= 0+0t+0t^2).$$

Then we get

$$(a+b+c)+(-3a-3c)t+(b+c)t^2 = 0,$$

or

$$a+b+c = 0, -3a-3c = 0, b+c = 0.$$

You can check that this system has the unique solution $a = b = c = 0$, so the set is linearly independent.

We can argue now that the set also spans, using the Basis Theorem (you could also show it directly, or take coordinates with respect to your favorite basis and show that those vectors span). We know that $\{1, t, t^2\}$ is a basis for $\mathbb{P}_2$, so $\dim \mathbb{P}_2 = 3$. That means that a 3 element set which is linearly independent is automatically a basis, and therefore spans. $\square$

(4) (a) Let $\mathscr{E} = \{1, t\}$ be the standard basis for $\mathbb{P}_1$. Find $[1+t]_{\mathscr{E}}$ and $[2-t]_{\mathscr{E}}$, and use this to explain why $\mathscr{B} = \{1+t, 2-t\}$ is a basis for $\mathbb{P}_1$.

Solution. $[1+t]_{\mathscr{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[2-t]_{\mathscr{E}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. The vectors $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ are a basis for $\mathbb{R}^2$ (how can you tell?), so $\mathscr{B}$ is a basis for $\mathbb{P}_1$. $\square$

(b) Let $\mathscr{B}$ be as in part (a), and let $\mathscr{C} = \{1-t, t\}$ be another basis. Find $[2t+2]_{\mathscr{B}}$ and $[2t+2]_{\mathscr{C}}$.

Solution. We have $2t+2 = 2(1+t)+0(2-t)$, and we also have $2t+2 = 2(1-t)+4t$. Thus

$$[2t+2]_{\mathscr{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad [2t+2]_{\mathscr{C}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

□