LINEAR ALGEBRA WORKSHEET 1

MATH1014 SPRING SESSION

(1) Find the angle that $\mathbf{v} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$ makes with: (a) the positive x-axis, (b) the positive y-axis, and (c) the positive z-axis. Is the angle acute ($0 < \theta < \pi/2$), right ($\theta = \pi/2$), or obtuse ($\pi/2 < \theta < \pi$)?

Solution. (a) $\cos(\theta_x) = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{8}{9}$ so $\theta_x = \arccos(8/9)$. Since $\cos(\theta_x) > 0$, the angle is acute.
(b) $\cos(\theta_y) = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|} = \frac{-1}{9}$ so $\theta_y = \arccos(-1/9)$. Since $\cos(\theta_y) < 0$, the angle is obtuse.
(c) $\cos(\theta_z) = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|} = \frac{4}{9}$ so $\theta_z = \arccos(4/9)$. Since $\cos(\theta_z) > 0$, the angle is acute. □

(2) (a) Check whether the two lines in $\mathbb{R}^2$ represented by the vector equations

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are parallel. If they are not, check if/where they intersect.
(b) Do the same for the following two lines in $\mathbb{R}^3$:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Solution. (a) The lines are not parallel because $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are not multiples of each other. To find the point of intersection, we look for s and t which solve

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} + s\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We have used different variables s and t because there is no reason that the intersection point has to correspond to the same t-value on the two lines. Re-arranging, this is equivalent to solving

$$\begin{bmatrix} 4t - s \\ 5t - 2s \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

This is a system of two equations in two variables. It has a unique solution $t = 2/3$ and $s = -1/3$. Plugging either value in to the line gives the intersection point $\begin{bmatrix} 11/3 \\ 16/3 \end{bmatrix}$.
(b) Again the lines are not parallel. Solving

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} + s\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

yields a unique solution $t = 1$ and $s = 1/2$, which gives a point of intersection $\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$. □

(3) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
(a) Find a vector in $\mathbb{R}^3$ which is orthogonal to $\mathbf{v}$.

(b) Compute the vector and scalar projection of $\mathbf{v}$ onto $\mathbf{u}$.

(c) Find an equation for the plane parallel to $\mathbf{v}$ and $\mathbf{u}$ containing the point $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Solution. (a) There are lots and lots of correct answers! One correct answer is the normal vector used in part (c). Some other correct answers are $\begin{bmatrix} 0 \\ 47 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} t \\ 0 \\ -1 \end{bmatrix}$.

(b)

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

$$\operatorname{comp}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|u\|} = \frac{1}{\sqrt{2}}.$$

Note that the scalar projection is the sign of $\mathbf{u} \cdot \mathbf{v}$ times the length of the vector projection.

(c) We need to find a vector $\mathbf{n}$ that is normal (orthogonal) to the plane. For this we can use

$$\mathbf{n} = \mathbf{v} \times \mathbf{u} = \begin{bmatrix} -t \\ t \\ 1 \end{bmatrix}.$$

Then the plane consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$$\mathbf{n} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right) = 0.$$

Expanding this gives

$$-t(x-3) + t(y-2) + (z-1) = 0.$$

□

(4) Are the lines

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$$

the same?

Solution. Yes they are! First notice that the lines are parallel, because the directions $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$ are multiples of each other. So the lines must be the same if they have at least one point of intersection. You can find a point of intersection using the method from exercise 2. □