LINEAR ALGEBRA WORKSHEET 2

MATH1014 SPRING SESSION

(1) (a) Find a vector $\mathbf{v}_0$ on the plane described by the equations $2x + y - z = 4$, and write the equation in the form $\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{v}_0$.

(b) Find the distance between the point $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and the plane in part (a).

Solution. (a) We need to find any x,y, and z satisfying that equation - there are lots of choices! For example $\mathbf{v}_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Then with $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, the equation becomes $\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{v}_0$.

(b) With $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, our formula gives the distance as

$$\frac{|\mathbf{n} \cdot (\mathbf{u} - \mathbf{v}_0)|}{\|\mathbf{n}\|} = \frac{|-3|}{\sqrt{6}} = \frac{3}{\sqrt{6}}.$$

□

(2) Is the given set H a subspace of the vector space V?
(a) $V = \mathbb{P}_3$, H is all polynomials of the form $p(x) = a(x-1)^2 + bx$
(b) $V = \mathbb{P}_3$, H is all polynomials of the form $p(x) = ax^2 - 1 + bx$
(c) $V = M_{2\times 2}$, H is all non-invertible 2×2 matrices

Solution. (a) Yes, it is $\mathrm{Span}\{(x-1)^2, x\}$.

(b) No. Every polynomial in H has a non-zero constant term, so the zero polynomial is not in H.

(c) No. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in H but their sum

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is invertible and therefore not in H. □

(3) Are these subspaces of $\mathbb{R}^3$? If so, write them as either Col$A$ or Nul$A$ for some matrix A.

(a) vectors of the form $\begin{bmatrix} s+3t+2 \\ s-2t \\ t \end{bmatrix}$

(b) vectors of the form $\begin{bmatrix} s-2t \\ s+t \\ t \end{bmatrix}$

(c) vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x - y = y - z$ and $y + 2x = z$

Solution. (a) Not a subspace; does not contain the zero vector.

(b) These are vectors of the form

$$s\begin{bmatrix}1\\1\\0\end{bmatrix}+t\begin{bmatrix}-2\\1\\1\end{bmatrix},$$

so this is $\mathrm{Col}\begin{bmatrix}1&-2\\1&1\\0&1\end{bmatrix}$. (This is the "most straightforward" matrix, but for example another choice would be $\begin{bmatrix}1&-2&2\\1&1&2\\0&1&0\end{bmatrix}$.)

(c) These equations are equivalent to $x-2y+z=0$ and $2x+y-z=0$. A vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ satisfies these equations exactly when

$$\begin{bmatrix}1&-2&1\\2&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

So this space is $\mathrm{Nul}\begin{bmatrix}1&-2&1\\2&1&-1\end{bmatrix}$. Again, there are many more choices possible! □

(4) Think of the functions e^x, x^2, and $x+2$ as vectors in $\mathscr{F}$, the vector space of all functions on $\mathbb{R}$. Find a function f in $\mathrm{Span}\{e^x,x^2,x+2\}$ such that $f(0)=1$ and $f(1)=2$.

Solution. A function f in $\mathrm{Span}\{e^x,x^2,x+2\}$ is given by the rule $f(x)=ae^x+bx^2+c(x+2)$ for some scalars a,b,c. Plugging in, we find that the equations $f(0)=1$ and $f(1)=2$ impose the requirement that $a+2c=1$ and $ae^1+b+3c=2$. We just need to find one solution to these equations. For example, one solution is $c=0$, $a=1$, and $b=2-e$. This gives us a function $f(x)=e^x+(2-e)x^2$ satisfying the given property. □