LINEAR ALGEBRA WORKSHEET 4

MATH1014 SPRING SESSION

(1) $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \right\}$ and $\mathscr{D} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ are two bases for $\mathbb{R}^3$.

(a) Find the change of basis matrix $\underset{\mathscr{D} \leftarrow \mathscr{B}}{P}$.

Solution. If $\mathscr{E}$ is the standard basis, we have

$$\underset{\mathscr{E} \leftarrow \mathscr{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad \underset{\mathscr{E} \leftarrow \mathscr{D}}{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix we want is given by

$$\underset{\mathscr{D} \leftarrow \mathscr{B}}{P} = \underset{\mathscr{D} \leftarrow \mathscr{E}}{P} \, \underset{\mathscr{E} \leftarrow \mathscr{B}}{P} = \left(\underset{\mathscr{E} \leftarrow \mathscr{D}}{P} \right)^{-1} \underset{\mathscr{E} \leftarrow \mathscr{B}}{P}.$$

We compute this using the row reduction method.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 & 2 & 4 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -3 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 4 \end{array} \right].$$

So

$$\underset{\mathscr{D} \leftarrow \mathscr{B}}{P} = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}.$$

□

(b) If $\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$, find the coordinates of $\mathbf{x}$ relative to $\mathscr{D}$.

Solution. From inspection we see that $[\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, so

$$[\mathbf{x}]_{\mathscr{D}} = \underset{\mathscr{D} \leftarrow \mathscr{B}}{P} [\mathbf{x}]_{\mathscr{B}} = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix}.$$

□

(2) Let $\mathscr{B} = \{1 + t, 2 - t\}$ and $\mathscr{C} = \{1 - t, t\}$ be two bases for $\mathbb{P}_1$.

(a) Express $2t + 2$ in $\mathscr{B}$ and $\mathscr{C}$ coordinates.

Solution. $2t + 2 = 2(1 + t) + 0(2 - t)$ and $2t + 2 = 2(1 - t) + 4t$, so

$$[2t + 2]_{\mathscr{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad [2t + 2]_{\mathscr{C}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

□

(b) Is it easier to express the $\mathscr{B}$ basis vectors in $\mathscr{C}$ coordinates, or the $\mathscr{C}$ basis vectors in $\mathscr{B}$ coordinates? Why might you want to know this before you start the next problems?

Solution. Because $\mathscr{C}$ has a very simple vector like t in it, it is easier to express $\mathscr{B}$ basis vectors in terms of $\mathscr{C}$ basis vectors than vice versa (if you disagree, try both!). That means it's easier to find $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$ rather than $\underset{\mathscr{B}\leftarrow\mathscr{C}}{P}$. Good thing the next two problems appear in the order that they do! $\square$

(c) Find the change of basis matrix from $\mathscr{B}$ to $\mathscr{C}$ coordinates.

Solution. $1+t=(1-t)+2t$ and $2-t=2(1-t)+t$, so

$$[1+t]_{\mathscr{C}}=\begin{bmatrix}1\\2\end{bmatrix}, \qquad [2-t]_{\mathscr{C}}=\begin{bmatrix}2\\1\end{bmatrix}.$$

So we have

$$\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}=\begin{bmatrix}1&2\\2&1\end{bmatrix}.$$

$\square$

(d) Find the change of basis matrix from $\mathscr{C}$ to $\mathscr{B}$ coordinates.

Solution.

$$\underset{\mathscr{B}\leftarrow\mathscr{C}}{P}=\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}^{-1}=\begin{bmatrix}1&2\\2&1\end{bmatrix}^{-1}=\begin{bmatrix}-1/3&2/3\\2/3&-1/3\end{bmatrix}.$$

$\square$

(3) Suppose $\mathscr{B}$ and $\mathscr{C}$ are two bases for $M_{2\times3}$. How many rows and columns does the change of basis matrix $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$ have?

Solution. A basis for $M_{2\times3}$ is

$$\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}.$$

So $\dim M_{2\times3}=6$. Thus $\underset{\mathscr{C}\leftarrow\mathscr{B}}{P}$ is a 6×6 matrix. $\square$

(4) (a) A person gets coffee every morning at one of two cafes, called (conveniently) Cafe 1 and Cafe 2. They notice that if they get coffee at Cafe 1 one morning, then the next morning they get coffee at Cafe 1 20% of the time, and at Cafe 2 80% of the time. On the other hand, if they get coffee at Cafe 2 one morning, then the next morning they get coffee at Cafe 1 60% of the time, and at Cafe 2 40% of the time. In the long run, what percentage of the time will they go to Cafe 1, and what percentage of the time will they go to Cafe 2?

Solution. The transition matrix which corresponds to this system is

$$T=\begin{bmatrix}.2&.6\\.8&.4\end{bmatrix},$$

where the first row/column is Cafe 1, and the second row/column is Cafe 2. We row reduce $T-I$ to get

$$\begin{bmatrix}1&-3/4\\0&0\end{bmatrix}$$

which means that the general solution to the homogeneous equation is $x_1=3/4x_2$, or in vector form

$$x_2\begin{bmatrix}3/4\\1\end{bmatrix}.$$

The only probability vector which is a solution corresponds to $x_2 = 4/7$, which gives a steady state of

$$\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}.$$

In other words, the person goes to Cafe 1 about 43% of the time, and Cafe 2 about 57% of the time. $\square$

(b) If a coffee costs \$3 at Cafe 1 and \$4 at Cafe 2, what is their long-term average daily coffee cost?

Solution. In the long run, the person will go to the two cafes according to the probabilities from part (a), so the expected cost is

$$\$3 \cdot 3/7 + \$4 \cdot 4/7 \approx \$3.57$$

$\square$