LINEAR ALGEBRA WORKSHEET 5

MATH1014 SPRING SESSION

(1) Find all eigenvalues of the matrix. Is the matrix diagonalisable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Solution. To find the eigenvalues we find the characteristic polynomial and solve the characteristic equation. We have

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} \\ &= -\lambda(-2\lambda + \lambda^2 + 1) + (-\lambda + 1) - (-1 + 2 - \lambda) \\ &= -\lambda^3 + 2\lambda^2 - \lambda \\ &= -\lambda(\lambda^2 - 2\lambda + 1) \\ &= -\lambda(\lambda - 1)^2, \end{aligned}$$

so the eigenvalues are $\lambda = 0$ (with algebraic multiplicity 1) and $\lambda = 1$ (with algebraic multiplicity 2).

We now investigate the eigenspaces. First we look at $E_1 = \operatorname{Nul}(A - I)$. We can row reduce

$$A - I = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has two free variables, so we can conclude that $\dim \operatorname{Nul}(A - I) = 2$, which is the same as the multiplicity. Since $\lambda = 0$ has multiplicity 1, and every eigenvalue has at least one eigenvector, we will have $\dim \operatorname{Nul}(A) = 1$, and we can conclude that A is diagonalisable.

In order to diagonalise A, we will need to find an eigenvector basis for A. We first find a basis for $\operatorname{Nul}(A - I)$. From above, we see that vectors in $\operatorname{Nul}(A - I)$ are the ones solving the equation $x_1 + x_2 + x_3 = 0$. A basis for this space is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Next we find a basis for $\operatorname{Nul}(A)$. We row reduce

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This corresponds to the system $x_1 = x_3$ and $x_2 = -x_3$. The general form of the solution is

$$\begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

which gives $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$ as a basis for Nul(A). Putting this all together, we have an eigenvector basis

$$\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

corresponding to the eigenvalues 1, 1, and 0 respectively. We conclude that

$$A = PDP^{-1}$$

with

$$P = \begin{bmatrix}-1 & -1 & 1\\ 1 & 0 & -1\\ 0 & 1 & 1\end{bmatrix}, \qquad D = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

□

(2) (a) Suppose that A is diagonalisable and has only one eigenvalue λ. Explain why $A = \lambda I$.

Solution. If A is diagonalisable, we can write $A = PDP^{-1}$ where the columns of P are an eigenvector basis for A, and D is a diagonal matrix with the corresponding eigenvalues. Since λ is the only eigenvalue, we have $D = \lambda I$. Then

$$A = PDP^{-1} = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I.$$

□

(b) Is the matrix

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$$

diagonalisable?

Solution. No it is not. Since the matrix is triangular, the eigenvalues are the diagonal entries, and we can conclude that 1 is the only eigenvalue. By part (a), the only diagonalisable matrix with only the eigenvalue 1 is the identity. Since this matrix is not the identity, it must not be diagonalisable. □

(3) Which of the following are possible for a 3×3 matrix? If it's impossible, explain why. If it's possible, give an example.

(a) A has exactly two distinct eigenvalues and is diagonalisable

Solution. This is possible, e.g.

$$A = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

has the eigenvalues 0 and 2, and is diagonalisable (with the standard basis an eigenvector basis). □

(b) B has exactly four distinct eigenvalues and is diagonalisable

Solution. This is impossible. A 3×3 matrix has at most 3 distinct eigenvalues. □

(c) C has exactly three distinct eigenvalues and is not diagonalisable

Solution. This is impossible. If a 3×3 matrix has three distinct eigenvalues, then it is diagonalisable. Each eigenvalue has a corresponding eigenvector, and putting these three together gives an eigenvector basis. $\square$

(d) D has exactly two distinct eigenvalues and is not invertible.

Solution. A matrix is invertible exactly when 0 is not an eigenvalue. So the example for part (a) works for this one also. Note: invertibility and diagonalisability are not directly connected. You should convince yourself of this by showing that any of the four combinations of invertible/non-invertible and diagonalisable/non-diagonalible are possible. $\square$

(4) For which values of t does the matrix

$$\begin{bmatrix} 1 & t \\ 2 & 3 \end{bmatrix}$$

have two distinct eigenvalues? (Hint: a quadratic equation $ax^2 + bx + c = 0$ has two different real solutions exactly when $b^2 > 4ac$).

Solution. The characteristic equation is

$$0 = \det \begin{bmatrix} 1-\lambda & t \\ 2 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + (3-2t).$$

By the hint, this equation has two distinct solutions exactly when

$$16 > 4(3-2t) = 12 - 8t.$$

That is, when $8t > -4$, or $t > -1/2$.

As an aside, the matrix will be diagonalisable for these values of t. Can you explain why it won't be diagonalisable for any other values of t? $\square$