Math 110, Fall 2012, Sections 109-110
Worksheet 10

1. True or false:

   (a) If A and B are row equivalent, then A is diagonalizable if and only if B is diagonalizable.
   (b) If A is diagonalizable, then $\dim E_\lambda$ is equal to the multiplicity of λ for all eigenvalues λ.
   (c) The multiplicities of all eigenvalues of a matrix add up to the size of the matrix

   Solution:

   (a) False. Consider
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

   (b) True by Theorem 5.9(a).

   (c) False, since a matrix need not have eigenvalues. The the characteristic polynomial splits, then this is true since the sum of the multiplicities would be the degree of the characteristic polynomial.

2. Are the following matrices diagonalizable? Try to answer without taking a single determinant or doing a single row operation.
$$A = \begin{pmatrix} -99 & 42 & 16 \\ 0 & e & -12 \\ 0 & 0 & 432 \end{pmatrix}, \qquad B = \begin{pmatrix} \pi & 0 & 0 \\ 0 & 47 & 29 \\ 0 & 0 & 47 \end{pmatrix}.$$

   Solution: A is diagonalizable since it has 3 distinct eigenvalues. B is not diagonalizable, since $\operatorname{rank} B - 47I = 2$, and so $\dim E_{47} = 1 < 2 = m_{47}$.

3. Let
$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
   Show that A is not diagonalizable when considered as an element of $M_{2\times 2}(\mathbb{R})$, but is diagonalizable as an element of $M_{2\times 2}(\mathbb{C})$.

   Solution: The characteristic polynomial of R is $p(t) = t^2 + 1$. Since $p(t)$ has no roots over $\mathbb{R}$, R is not diagonalizable. Over $\mathbb{C}$, $p(t)$ has two distinct roots $\pm i$, so R is diagonalizable.

4. Let R be as above. Find real numbers a_0, a_1, a_2 such that $a_0 I + a_1 R + a_2 R^2 = 0$.

   Solution: By the Cayley-Hamilton theorem, $R^2 + I = 0$.

5. Find a basis for the T-cyclic subspace of R^3 generated by v, and where $v = (1, 0, 0)$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T(x, y, z) = (x - y + z, x + 2y - z, 3z).$$

   Find the characteristic polynomial of T restricted to this subspace.

   Solution: $T(v) = (1, 1, 0)$ and $T^2(v) = (0, 3, 0)$. Thus $\{v, T(v)\}$ is linearly independent but $T^2(v) - 3T(v) + 3v = 0$. Now Theorem 5.22 says that the characteristic polynomial of T restricted to span$\{v, T(v)\}$ is $t^2 - 3t + 3$.