Math 110, Fall 2012, Sections 109-110
Worksheet 11

1. Do the following definitions give inner products on the vector space V?
   (a) $V = \mathbb{R}^2$, $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + 3x_2y_2$.
   (b) $V = \mathbb{R}^2$, $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_2 + x_2y_1$
   (c) $V = P_2(\mathbb{C})$, $\langle p, q \rangle = p(0)\overline{q(0)} + p(4)\overline{q(4)} + p(47)\overline{q(47)}$

2. (a) If V is an inner product space, what does it mean for a pair of vectors in V to be orthogonal? What does it mean for a subset $S \subset V$ to be orthogonal? What does it mean for S to be orthonormal?
   (b) Prove that if x and y are orthogonal, then $\|x + y\| = \sqrt{\|x\|^2 + \|y\|^2}$. What geometric fact is this when $V = \mathbb{R}^2$ with the standard inner product?

3. (a) Suppose $v, w \in V$ are non-zero vectors. Define
$$y = \frac{\langle v, w \rangle}{\langle w, w \rangle} w, \qquad z = v - \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$
   Show that
   i.) $v = y + z$
   ii.) $y \in \text{span}\{w\}$ and z is orthogonal to every element of $\text{span}\{w\}$.
   (b) Draw a picture that demonstrates part (a) when $V = \mathbb{R}^2$, $v = (1, 1)$ and $w = (2, 1)$.

4. (a) Suppose that V is a finite-dimensional inner product space and that $\{x_1, \dots, x_n\}$ is an orthonormal basis for V. If $x = c_1x_1 + \cdots + c_nx_n$, what is a simple formula for c_j?
   (b) Give a simple formula for $x_i^* \in V^*$.
   (c) (Bonus) Show that if $f \in V^*$, then there exists $y \in V$ such that $f(x) = \langle x, y \rangle$ for all $x \in V$.