Math 110, Fall 2012, Sections 109-110
Worksheet 11

1. Do the following definitions give inner products on the vector space V?

   (a) $V = \mathbb{R}^2$, $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + 3x_2y_2$.

   (b) $V = \mathbb{R}^2$, $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_2 + x_2y_1$

   (c) $V = P_2(\mathbb{C})$, $\langle p, q \rangle = p(0)\overline{q(0)} + p(4)\overline{q(4)} + p(47)\overline{q(47)}$

   Solution: (a) Yes, (b) No, (c) Yes. Note that if a polynomial of degree at most 2 has three roots, then it is the zero polynomial.

2. (a) If V is an inner product space, what does it mean for a pair of vectors in V to be orthogonal? What does it mean for a subset $S \subset V$ to be orthogonal? What does it mean for S to be orthonormal?

   (b) Prove that if x and y are orthogonal, then $\|x + y\| = \sqrt{\|x\|^2 + \|y\|^2}$. What geometric fact is this when $V = \mathbb{R}^2$ with the standard inner product?

   Solution:

   (a) Orthogonal means having inner product equal to zero. A set is called orthogonal if any pair of elements are orthogonal. An orthonormal set is an orthogonal set S such that $\langle x, x \rangle = 1$ for all $x \in S$.

   (b) We have

   $$\|x + y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle = \langle y, y \rangle = \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2 .$$

   This is the Pythagorean Theorem.

3. (a) Suppose $v, w \in V$ are non-zero vectors. Define

   $$y = \frac{\langle v, w \rangle}{\langle w, w \rangle} w, \qquad z = v - \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

   Show that

   i.) $v = y + z$

   ii.) $y \in \operatorname{span}\{w\}$ and z is orthogonal to every element of $\operatorname{span}\{w\}$.

(b) Draw a picture that demonstrates part (a) when $V = \mathbb{R}^2$, $v = (1,1)$ and $w = (2,1)$.

Solution:

(a) Part (i) is clear from the definitions. It is also clear that $y \in \operatorname{span}\{w\}$. It suffices to check that $\langle z, w\rangle = 0$ to complete part (ii). We have

$$\langle z, w\rangle = \langle v, w\rangle - \langle \frac{\langle v, w\rangle}{\langle w, w\rangle} w, w\rangle = \langle v, w\rangle - \frac{\langle v, w\rangle}{\langle w, w\rangle}\langle w, w\rangle = 0.$$

4. (a) Suppose that V is a finite-dimensional inner product space and that $\{x_1, \dots, x_n\}$ is an orthonormal basis for V. If $x = c_1x_1 + \cdots + c_nx_n$, what is a simple formula for c_j?

   (b) Give a simple formula for $x_i^* \in V^*$.

   (c) (Bonus) Show that if $f \in V^*$, then there exists $y \in V$ such that $f(x) = \langle x, y\rangle$ for all $x \in V$.

Solution: (a) We have

$$\langle x, x_j\rangle = c_1\langle x_1, x_j\rangle + \cdots + c_k\langle x_n, x_j\rangle = c_j.$$

(b) It follows from (a) and the definition of x_i^* that $x_i^*(x) = \langle x, x_i\rangle$.

(c) Given $f \in V^*$, set $y = \overline{f(x_1)}x_1 + \cdots + \overline{f(x_n)}x_n$. Define $g(x) = \langle x, y\rangle$. We wish to show that $g(x) = f(x)$ for all $x \in V$, and it suffices to check it on basis vectors x_j. We have

$$g(x_j) = \langle x_j, \overline{f(x_1)}x_1 + \cdots + \overline{f(x_n)}x_n\rangle = f(x_1)\langle x_j, x_1\rangle + \cdots + f(x_n)\langle x_j, x_n\rangle = f(x_j)$$

as desired.