Math 110, Fall 2012, Sections 109-110
Worksheet 12

1. True or false? Assume V is a finite-dimensional inner product space over $\mathbb{C}$ and that T is a linear operator on V.

   (a) If T is self-adjoint and β is a basis for V, then $[T]_\beta$ is self-adjoint.

   (b) If x is an eigenvector for T, then x is also an eigenvector for T^*.

   (c) If T and S are self-adjoint, then $T + S$ is self-adjoint.

   (d) If T is self-adjoint and $c \in F$, then cT is self-adjoint

   (e) If T and S are normal, then $T + S$ is normal.

   (f) If T is normal and $c \in F$, then cT is normal.

   Solution:

   (a) False. This becomes true if β is assumed to be orthonormal. If not, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

   so that L_A is self-adjoint. However, $[L_A]_\beta$ is not self-adjoint where $\beta = \{(1, 0), (1, 1)\}$.

   (b) False. If $T(x, y) = (x + y, y)$. Then $T(1, 0) = (1, 0)$, but $T^*(1, 0) = (1, 1)$.

   (c) True, since $(S + T)^* = S^* + T^*$.

   (d) False. If $T \neq 0$ and $F = \mathbb{C}$, then $(iT)^* = -iT^* = -iT$ is not self-adjoint.

   (e) False. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   Then L_A is self-adjoint and L_B is skew-adjoint. In particular, both are normal. But

$$A + B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

   does not commute with its adjoint.

   (f) True, as

$$(cT)^*(cT) = |c|^2 T^*T = |c|^2 TT^* = (cT)(cT)^*.$$

2. Let T be a normal operator on a complex finite-dimensional inner product space V. Prove that T is self-adjoint if and only if all the eigenvalues of T are real numbers.

   Solution: There exists an orthonormal basis β such that $[T]_\beta$ is a diagonal matrix with real entries. Thus
$$[T^*]_\beta = [T]_\beta^* = [T]_\beta,$$
   and so $T^* = T$.

3. Suppose that $A \in M_{n\times n}(\mathbb{C})$. Prove that $A^*A = I_n$ if and only if the columns of A form an orthonormal basis for $\mathbb{C}^n$.

   Solution: If the columns of A are $a_1, \ldots, a_n$, then
$$\begin{aligned} A^*A &= \begin{pmatrix} a_1^* \\ \vdots \\ a_n^* \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \\ &= \begin{pmatrix} a_1^*a_1 & \cdots & a_1^*a_n \\ \vdots & & \vdots \\ a_n^*a_1 & \cdots & a_n^*a_n \end{pmatrix} \\ &= \begin{pmatrix} \langle a_1, a_1\rangle & \cdots & \langle a_n, a_1\rangle \\ \vdots & & \vdots \\ \langle a_1, a_n\rangle & \cdots & \langle a_n, a_n\rangle \end{pmatrix}, \end{aligned}$$
   where the inner product is the standard one on $\mathbb{C}^n$. This matrix is the identity if and only if $\langle a_i, a_j\rangle = \delta_{ij}$. That is, unless the columns form an orthonormal set.

4. Let T be a self-adjoint linear operator on an inner product space V. Suppose W is a T-invariant subspace of V. Prove that $W^\perp$ is also T-invariant.

   Solution: Suppose $x \in W^\perp$, and we wish to show that $T(x) \in W^\perp$. So for all $y \in W$ we have
$$\langle T(x), y\rangle = \langle x, T(y)\rangle = 0$$
   since $y \in W$ and W is T-invariant. Thus $T(x) \in W^\perp$, which was to be shown.