Math 110, Fall 2012, Sections 109-110
Worksheet $12\frac{1}{2}$

1. Let V be a real inner product space.

   (a) (The Polarization Identity) Prove that for all $x, y \in V$ we have
   $$\langle x, y\rangle = \frac{1}{2}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

   (b) Prove that if U is a linear operator on V such that $\|Ux\| = \|x\|$ for all $x \in V$, then U is unitary. (Informally, this exercise says that "If a linear operator preserves lengths, then it also preserves angles." A similar exercise can be done for complex inner product spaces, but the complex version of (a) has more terms. See Exercise 6.1.20(b))

2. (The Cartesian Decomposition) Prove that if T is a linear operator on a finite-dimensional, complex inner product space V, then there exist unique self-adjoint operators A and B such that $T = A + iB$. Hint: how did we write any matrix as the sum of a symmetric and a skew-symmetric matrix? (This is an operator version of the fact that complex numbers can be written as $x + iy$ with x and y real numbers.)

3. (Positive operators and square roots) A self-adjoint operator A on an inner product space V is called *positive semi-definite* if $\langle Ax, x\rangle \geq 0$ for all $x \in V$. In the following, assume V is finite-dimensional.

   (a) If T is any linear operator on V, prove that T^*T is positive semidefinite.

   (b) Prove that if A is self-adjoint, then A is positive semidefinite if and only if all of its eigenvalues are non-negative real numbers (i.e. real numbers $\lambda \geq 0$)

   (c) Prove that if A is positive semidefinite, then there exists a unique positive semidefinite operator B such that $B^2 = A$. (Informally, this proves that "positive operators have unique positive square-roots." One can therefore talk unambiguously about $A^{\frac{1}{2}}$ if A is positive semi-definite.)

4. (Polar decomposition) Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Define the absolute value of T by $|T| = (T^*T)^{\frac{1}{2}}$, which makes sense by the previous exercise.

(a) Prove that $\|Tx\| = \||T|\,x\|$ for all $x \in V$.

(b) Prove that if T is invertible, then there exists a unique unitary operator U such that $T = U\,|T|$. (This is an analog of the fact that non-zero complex numbers can be written $z = e^{i\theta} r$ where $r = (\overline{z}z)^{\frac{1}{2}}$ is positive.)