Math 110, Fall 2012, Sections 109-110
Worksheet $12\frac{1}{2}$

1. Let V be a real inner product space.

   (a) (The Polarization Identity) Prove that for all $x, y \in V$ we have

   $$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

   (b) Prove that if U is a linear operator on V such that $\|Ux\| = \|x\|$ for all $x \in V$, then U is unitary. (Informally, this exercise says that "If a linear operator preserves lengths, then it also preserves angles." A similar exercise can be done for complex inner product spaces, but the complex version of (a) has more terms. See Exercise 6.1.20(b))

   Solution:

   (a) Observe that

   $$\|x+y\|^2 - \|x-y\|^2 = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle = 2\langle x, y\rangle + 2\langle y, x\rangle = 4\langle x, y\rangle$$

   since $F = \mathbb{R}$.

   (b) We have

   $$4\langle Ux, Uy\rangle = \|U(x+y)\|^2 - \|U(x-y)\|^2 = \|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$$

   for all $x, y \in V$, so U is unitary.

2. (The Cartesian Decomposition) Prove that if T is a linear operator on a finite-dimensional, complex inner product space V, then there exist unique self-adjoint operators A and B such that $T = A + iB$. Hint: how did we write any matrix as the sum of a symmetric and a skew-symmetric matrix? (This is an operator version of the fact that complex numbers can be written as $x + iy$ with x and y real numbers.)

   Solution: Note that $A = \frac{1}{2}(T + T^*)$ and $B = -\frac{i}{2}(T - T^*)$ are self-adjoint, and that $T = A + iB$.

   To prove uniqueness, suppose $T = A' + iB'$ as well. Then $0 = (A - A') + i(B - B')$. Taking adjoints yields $0 = (A - A') - i(B - B')$, and adding the two equations gives $2(A - A') = 0$. Thus $A = A'$. Now substituting above yields $0 = i(B - B')$, so $B = B'$. It follows that the A and B given above are unique.

3. (Positive operators and square roots) A self-adjoint operator A on an inner product space V is called *positive semi-definite* if $\langle Ax, x\rangle \geq 0$ for all $x \in V$. In the following, assume V is finite-dimensional.

   (a) If T is any linear operator on V, prove that T^*T is positive semidefinite.

   (b) Prove that if A is self-adjoint, then A is positive semidefinite if and only if all of its eigenvalues are non-negative real numbers (i.e. real numbers $\lambda \geq 0$)

   (c) Prove that if A is positive semidefinite, then there exists a unique positive semidefinite operator B such that $B^2 = A$. (Informally, this proves that "positive operators have unique positive square-roots." One can therefore talk unambiguously about $A^{\frac{1}{2}}$ if A is positive semi-definite.)

Solution:

(a) We have $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$.

(b) If A is self-adjoint but has a negative eigenvalue λ, then if x is an eigenvector with eigenvalue λ we have

$$\langle Ax, x\rangle = \lambda\langle x, x\rangle < 0$$

so A is not positive semidefinite.

Conversely, suppose A has all non-negative eigenvalues. Since A is self-adjoint, it has an orthonormal basis $x_1, \ldots, x_n$ of eigenvectors with not necessarily distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Given $x \in V$, it an be written

$$x = c_1x_1 + \cdots + c_nx_n.$$

We then have

$$\langle Ax, x\rangle = \sum_{i,j=1}^{n} c_i\overline{c_j}\langle Ax_i, x_j\rangle = \sum_{i,j=1}^{n} c_i\overline{c_j}\lambda_i\langle x_i, x_j\rangle = \sum_{i=1}^{n} |c_i|^2 \lambda_i \geq 0$$

(c) If A is positive semidefinite, then we can write $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ where $\lambda_1, \ldots \lambda_k$ are the distinct (nonnegative) eigenvalues of A. That is, any $x \in V$ can be written uniquely as $v_1 + \cdots v_k$ with $v_i \in E_{\lambda_i}$. Define an operator S by

$$S(v_1 + \cdots + v_k) = \lambda_1^{\frac{1}{2}}v_1 + \cdots + \lambda_k^{\frac{1}{2}}v_k,$$

where the nonnegative square root of nonnegative real numbers is used. S is well-defined by the uniqueness of the decomposition, and linearity is straightforward to check.

If $\{x_i\}$ is any orthonormal basis of eigenvectors of A, then each x_i is also an eigenvector of S by construction, so S possess an orthonormal basis of eigenvectors. Thus S is normal, and by construction has all nonnegative eigenvalues, so S is positive semidefinite. We have

$$S^2(v_1 + \cdots + v_k) = \lambda_1 v_1 + \cdots + \lambda_k v_k = A(v_1 + \cdots + v_k),$$

so $S^2 = A$ is a positive semidefinite square root.

To prove uniqueness, suppose that we have some other positive operator S' such that $(S')^2 = A$. Since S' is positive semidefinite, there exists an orthonormal basis $y_1, \ldots, y_n$ such that $S'(y_j) = \mu_j y_j$ with $\mu_j \geq 0$. We then have

$$A(y_j) = (S')^2(y_j) = \mu_j^2 y_j.$$

So each y_j is an eigenvector of A, and thus an eigenvector of S (since the two were constructed to have the same eigenspaces). If $Sy_j = \mu y_j$, then

$$A(y_j) = S^2(y_j) = \mu^2(y_j).$$

Thus $\mu^2 = \mu_j^2$, and since both μ and μ_j are nonnegative, we have $\mu = \mu_j$. Thus $S(y_j) = S'(y_j)$. Since y_j was an arbitrary element of a basis, we must have $S = S'$, proving uniqueness.

4. (Polar decomposition) Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Define the absolute value of T by $|T| = (T^*T)^{\frac{1}{2}}$, which makes sense by the previous exercise.

   (a) Prove that $\|Tx\| = \||T|\,x\|$ for all $x \in V$.

   (b) Prove that if T is invertible, then there exists a unique unitary operator U such that $T = U\,|T|$. (This is an analog of the fact that non-zero complex numbers can be written $z = e^{i\theta} r$ where $r = (\overline{z}z)^{\frac{1}{2}}$ is positive.)

   Solution:

   (a) We have

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \langle |T|^2\, x, x \rangle = \langle |T|\, x, |T|\, x \rangle = \||T|\, x\|^2 .$$

(b). Since T is invertible, we have $N(T) = \{0\}$. But (a) says $T(x) = 0 \iff |T|(x) = 0$, so $N(|T|) = \{0\}$ as well. Thus $|T|$ is invertible. Let $U = T|T|^{-1}$. By construction, we have $T = U|T|$. It remains to show that U is unitary.

We will assume exercise 1(b) for complex inner product spaces, which wasn't proven. It is very similar to the real case. See the references exercise from the book. With that assumption, it suffices to prove that $\|Ux\| = \|x\|$ for all $x \in V$. Since U is unitary if and only if U^{-1} is unitary, we will actually prove $\|U^{-1}x\| = \|x\|$ for all x. We have

$$\left\|U^{-1}x\right\| = \left\||T|\,T^{-1}x\right\| = \left\|TT^{-1}x\right\| = \|x\|\,,$$

where in the second equality we used part (a). Thus U^{-1}, and therefore U, are unitary.

All that remains to prove is uniqueness, but that is is as if $T = U'|T|$, then it follows immediately that $U' = T|T|^{-1}$.