Math 110, Fall 2012, Sections 109-110
Worksheet 1

1. Let V be a vector space over a field F. Carefully prove the following statement using only the axioms of vectors spaces and fields: Suppose $a \in F$ and $b \in V$. Then $ab = 0$ if and only if $a = 0$ or $b = 0$.

2. Why is the following question ill-posed (i.e. why doesn't it make any sense): "Let V be the set of cows in California. Is V a vector space over $\mathbb{F}_7$?"

3. Are the following vector spaces over $\mathbb{R}$?

   (a) The set of all real 2×2 matrices of the form
   $$\begin{pmatrix} a & 2 \\ 2 & b \end{pmatrix}$$
   with the usual matrix addition and scalar multiplication.

   (b) The set of all real 2×2 matrices of the form
   $$\begin{pmatrix} a & a+b \\ a+b & b \end{pmatrix}$$
   with the usual matrix addition and scalar multiplication.

   (c) The set of all real 2×2 matrices of the form
   $$\begin{pmatrix} a^2 & a \\ b & 0 \end{pmatrix}$$
   with the usual matrix addition and scalar multiplication.

4. Consider the subspaces of $M_{n\times n}(F)$ consisting: W_0, consisting of traceless matrices, W_1 consisting of matrices (a_{ij}) with $a_{ii} = 0$ for all i, and W_2 consisting of all strictly upper triangular matrices. Prove that $W_2 \subseteq W_0 \cap W_1$.

5. Let $W_1 = \{(\alpha, \alpha) : \alpha \in \mathbb{R}\}$ and let $W_2 = \{(\beta, -\beta) : \beta \in \mathbb{R}\}$. Prove that $\mathbb{R}^2 = W_1 \oplus W_2$.

1) Suppose first that $ab = 0$. It suffices to prove that if $a \neq 0$, then $b = 0$. Since $a \neq 0$, there exists an element $a^{-1} \in F$ with $aa^{-1} = 1$. Multiplying both sides of $ab = 0$ by a^{-1} on the left, we get $a^{-1}(ab) = a^{-1}0$. By (VS6), we have $(a^{-1}a)b = a^{-1}0$. Since $a^{-1}a = 1$, (VS5) says that $b = a^{-1}0$. All that remains is to show that $a^{-1}0 = 0$. That will follow from the mini-problem:

Suppose $c \in F$ and 0 is the zero vector in some vector space over F. Then $c0 = 0$. Proof: By (VS3) and (VS7), we have $c0 = c(0 + 0) = c0 + c0$. Now adding $-(c0)$ (which exists by (VS4)) to both sides on the right gives $c0 + (-c0) = (c0 + c0) + (-c0)$. By (VS4) the left-side becomes 0, and by (VS2) and (VS4) the right side becomes $\mathrm{c}0 + 0$. Applying (VS3) now gives $\mathrm{c}0 = 0$.

Conversely, suppose $a = 0$ or $b = 0$, and we will show that $ab = 0$. We proceed in two cases. In the first case, suppose $b = 0$. Then $ab = a0 = 0$ by the mini-problem we proved in the previous part. The other case (proving $0b = 0$) is similar, and left to you.

2) It doesn't make sense to ask if a set is a vector space without specifying the operations.

3) All of these examples are subsets of $V = M_{2\times 2}(\mathbb{R})$ being given with the same operations as on V. A theorem from class says that such a subset is a vector space under those operations if and only if it is a subspace of V.

a) Not a subspace, and therefore not a vector space. It fails all three axioms, but it suffices to note that the zero matrix is not an element.

b) This is a subspace. Call it W_2. First, we check that W_2 contains the zero matrix. Observe that the zero matrix is of the given form, with $a = b = 0$. Next we take $C, D \in W_2$ and show that $C + D = W_2$. Since $C, D \in W_2$ we can write them as

$$C = \begin{pmatrix} a_1 & a_1 + b_1 \\ a_1 + b_1 & b_1 \end{pmatrix}, \qquad D = \begin{pmatrix} a_2 & a_2 + b_2 \\ a_2 + b_2 & b_2 \end{pmatrix}$$

for some real numbers a_1, a_2, b_1, and b_2. Then

$$C + D = \begin{pmatrix} a_1 + a_2 & (a_1 + a_2) + (b_1 + b_2) \\ (a_1 + a_2) + (b_1 + b_2) & b_1 + b_2 \end{pmatrix}.$$

Thus $C + D$ is of the given form, with $a = a_1 + a_2$ and $b = b_1 + b_2$, and so $C + D \in W_2$.

Finally, let C be as above and let x be a real numbers. We must show that $xC \in W_2$. We have

$$xC = C = \begin{pmatrix} xa_1 & xa_1 + xb_1 \\ xa_1 + xb_1 & xb_1 \end{pmatrix}.$$

This is of the given form with $a = xa_1$ and $b = xb_1$. Thus $xC \in W_2$ and we have shown that W_2 is a subspace of V. By the note at the start, this means W_2 is a vector space with the given operations.

c) Not a subspace, and therefore not a vector space by the comment at the beginning. To see that it's not a subspace, note that

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is in the set, but $A + A$ is not.

4) Let $A = (a_{ij}) \in W_2$. Then we must show that $A \in W_0$ and $A \in W_1$. Since $A \in W_2$, we know $a_{ij} = 0$ whenever $i \geq j$. Thus, in particular $a_{ii} = 0$ for all i. By definition, this means $A \in W_1$. We also have

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} 0 = 0,$$

and so $A \in W_0$. Thus $A \in W_0 \cap W_1$, which was to be shown.

5) There are two parts: we must show that $W_1 \cap W_2 = \{0\}$ and that $W_1 + W_2 = \mathbb{R}^2$. To show two sets A and B are equal, it is often helpful to prove that $A \subseteq B$ and $B \subseteq A$. This is what we do here.

First we set out to prove $W_1 \cap W_2 = \{0\}$. Note that $\{0\} \subseteq W_1 \cap W_2$, since $0 \in W_1$ and $0 \in W_2$. Conversely, we must show that $W_1 \cap W_2 \subseteq \{0\}$. Let $x = (x_1, x_2) \in W_1 \cap W_2$. Then $x_1 = x_2$ because $x \in W_1$. But $x_1 = -x_2$ because $x \in W_2$. Adding equations gives $2x_1 = 0$, and thus $x_1 = 0$. Since $x_1 = x_2$, this means $x_2 = 0$ so $x = (0, 0)$. Thus $x \in \{0\}$, which completes the proof that $W_1 \cap W_2 = \{0\}$.

We now have to show that $W_1 + W_2 = \mathbb{R}^2$. As before, we show that both sides are subsets of each other. The easy direction first:

We first prove $W_1 + W_2 \subseteq \mathbb{R}^2$. Let $x \in W_1 + W_2$. By definition, there exist $y \in W_1$ and $z \in W_2$ with $y + z = x$. Since $W_i \subseteq \mathbb{R}^2$, we know $y, z \in \mathbb{R}^2$. Thus $z = y + z \in \mathbb{R}^2$, and we have shown that $W_1 + W_2 \subseteq \mathbb{R}^2$.

Conversely, we must also prove that $\mathbb{R}^2 \subseteq W_1 + W_2$. Let $x = (x_1, x_2) \in \mathbb{R}^2$. We must find elements $y \in W_1$ and $z \in W_2$ with $y + z = x$ to show that $x \in W_1 + W_2$.

[Parenthetical remark: we'd now go off on scratch paper and try to figure out what those are. I'll show you my "scratch paper" after the proof]

Indeed, if

$$y = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_2)\right), \qquad z = \left(\frac{1}{2}(x_1 - x_2), \frac{1}{2}(x_2 - x_1)\right)$$

then $y \in W_1$ and $z \in W_2$. We also have $y + z = x$, and so $x \in W_1 + W_2$. Thus we have shown that $\mathbb{R}^2 \subseteq W_1 + W_2$, which completes the proof.

Now, you'd reasonably be asking where y and z came from. Notice that *how* I found them isn't part of the proof. But it's important to know where they came from, so here's my scratch paper that isn't part of the proof:

If I had elements $y = (\alpha, \alpha)$ and $z = (\beta, -\beta)$ then $x = y + z$ is equivalent to

$$\begin{aligned} \alpha + \beta &= x_1 \\ \alpha - \beta &= x_2. \end{aligned}$$

Using the techniques of Section 1.4, this linear system can be solved to get $\alpha = \frac{1}{2}(x_1 + x_2)$ and $\beta = \frac{1}{2}(x_1 - x_2)$.

To take away from this problem: notice how I showed that $A = B$ by showing $A \subseteq B$ and $B \subseteq A$. Each of those two subproblems can be solved by taking an arbitrary element in one set, and showing that it is in the other. The answer to 5 is long, but most of it is just technical things that you can get for free. This method of showings sets are equal isn't always the easiest way to do things, but it is a good thing to try. Other tricks will build off of this fundamental one.