Math 110, Fall 2012, Sections 109-110
Worksheet 3

1. Let V be a vector space over $\mathbb{R}$ and $\{v_1, v_2, \ldots, v_m\} \subset V$. List two or three differences between $\{v_1, v_2, \ldots, v_m\}$ and $\operatorname{span}\{v_1, v_2, \ldots, v_m\}$.

2. Let V be a vector space, and suppose $v_1, \ldots, v_k \in V$. What is $\dim \operatorname{span}\{v_1, \ldots, v_k\}$?

3. (a) I'm thinking of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. All I'll tell you is that $T(1, 1, 1) = (4, 7)$ and $T(1, 0, -1) = (-2, 3)$. Compute $T(4, 2, 0)$.

   (b) More generally, suppose if I have a linear transformation $T : V \to W$, and I tell you $T(v_1)$, $T(v_2)$, $\ldots$, and $T(v_k)$. For which $v \in V$ can you calculate $T(v)$?

4. Give an example of two vectors spaces V and W, and sets of vectors $\{v_1, v_2, v_3\} \subset V$, and $\{w_1, w_2, w_3\} \subset W$ such that there is *no* linear transformation with $T(v_i) = w_i$ for all i.

5. Suppose that T is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with $R(T) \subseteq N(T)$.

   (a) What are the possible values of $r(T)$?

   (b) What is $T(T(x))$ for $x \in \mathbb{R}^n$?

6. Suppose $T : V \to W$ is a linear transformation and that $\{v_1, \ldots, v_k\}$ spans V.

   (a) Give an example where $\{T(v_1), \ldots, T(v_k)\}$ does not span W.

   (b) Prove that if $R(T) = W$, then $\{T(v_1), \ldots, T(v_k)\}$ spans W.

1. Let $S = \{v_1, \ldots, v_m\}$. Then S is finite, while $\operatorname{span} S$ is infinite. Also, $\operatorname{span} S$ is a subspace of V while S is not.

2. We know that $\dim \operatorname{span}\{v_1, \ldots, v_k\} \leq k$, since some subset of $\{v_1, \ldots, v_k\}$ is a basis for its span. Any number between 0 and k is possible.

3. (a) Since $(4, 2, 0) = 2(1, 1, 1) + 2(1, 0 - 1)$, we have

$$T(4, 2, 0) = 2T(1, 1, 1) + 2T(1, 0, -1) = (4, 20).$$

(b) $T(v)$ can be calculated for any $v \in \text{span}\{v_1, \ldots, v_k\}$.

4. Take $V = W = \mathbb{R}^2$, $v_1 = (0,0)$, $w_1 = (1,0)$, and the other vectors to be anything you want. Linear transformations must take the zero vector to the zero vector, so no such linear transformation exists.

5. (a) Since $R(T) \subseteq N(T)$, we have $r(T) \leq n(T)$. The dimension theorem says that $n - n(T) = r(T) \leq n(T)$, so $r(T) \leq n/2$. We still must show that all values $0 \leq r(T) \leq n/2$ occur. Clearly 0 is a possible value, as the zero linear transformation satisfies $\{0\} \subseteq R(T) \subseteq N(T) = \mathbb{R}^n$. If $0 < k \leq n/2$, then the linear transformation

$$T(x_1, \ldots, x_n) = (0, \ldots, 0, x_1, \ldots, x_k)$$

has $R(T) \subseteq N(T)$ and $r(T) = k$. If S is the right-shift operator, then this $T = S^{n-k}$.

(b) For any $x \in \mathbb{R}^n$, we have $T(x) \in R(T)$ and thus $T(x) \in N(T)$. This means that $T(T(x)) = 0$.

6. (a) Take $V = W = R^2$, $\{v_1, v_2\}$ to be the standard basis, and $T(x) = 0$ for all $x \in \mathbb{R}^2$.

(b) We must show that given $y \in W$, it is in $\text{span}\{T(v_1), \ldots, T(v_k)\}$. Since T is surjective, there is some $x \in V$ with $T(x) = y$. Since $\{v_1, \ldots, v_k\}$ spans V, there are coefficients $c_j \in F$ with $x = c_1 v_1 + \cdots + c_k v_k$. Then we have

$$y = T(c_1 v_1 + \cdots + c_k v_k) = c_1 T(v_1) + \cdots + c_k T(v_k) \in \text{span}\{T(v_1), \ldots, T(v_k)\},$$

as desired.