Math 110, Fall 2012, Sections 109-110
Worksheet 4

1. Let V and W be vector spaces with $\dim V = n$ and $\dim W = m$, and let β and γ be ordered bases for V and W, respectively. Suppose $v \in V$, and let $T : V \to W$ be a linear transformation.

   (a) In what space do the following expressions live?

   i. $T(v)$
   ii. $[v]_\beta$
   iii. $[T]_\beta^\gamma$
   iv. $[T]_\beta^\gamma [v]_\beta$
   v. $T(v)$

   (b) What is wrong with the following expressions?

   i. $T([v]_\beta)$
   ii. $[v]_\gamma$
   iii. $[T]_\beta^\gamma v$
   iv. $[T(v)]_\beta^\gamma$

2. Suppose V and W are vector spaces with bases $\beta = \{v_1, v_2, v_3\}$ and $\gamma = \{w_1, w_2\}$, respectively. Suppose

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

   (a) What is $T(v_1 + v_2 + v_3)$?

   (b) If $\beta' = \{v_1, v_2, v_1 + v_2 + v_3\}$, what is $[T]_{\beta'}^\gamma$? (You may assume that β' is a basis for V, but you should also know how to prove it)

3. Let $T : V \to V$ be a linear transformation. What is the relationship between $N(T)$ and $N(T^2)$? What is the relationship between $R(T)$ and $R(T^2)$?

4. Let V and W be vector spaces over a field F and let $\mathcal{L}(V, W)$ denote the set of linear transformations from V to W. How do we define addition and scalar multiplication in $\mathcal{L}(V, W)$? How would you show that $\mathcal{L}(V, W)$ is a vector space under these operations?