Math 110, Fall 2012, Sections 109-110
Worksheet 4

1. Let V and W be vector spaces over a field F with $\dim V = n$ and $\dim W = m$. Let β and γ be ordered bases for V and W, respectively. Suppose $v \in V$, and let $T : V \to W$ be a linear transformation.
   (a) In what spaces do the following expressions live?
      i. $T(v)$
      ii. $[v]_\beta$
      iii. $[T]_\beta^\gamma$
      iv. $[T]_\beta^\gamma [v]_\beta$
   (b) What is wrong with the following expressions?
      i. $T([v]_\beta)$
      ii. $[v]_\gamma$
      iii. $[T]_\beta^\gamma v$
      iv. $[T(v)]_\beta^\gamma$
2. Suppose V and W are vector spaces with bases $\beta = \{v_1, v_2, v_3\}$ and $\gamma = \{w_1, w_2\}$, respectively. Suppose
$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$
   (a) What is $T(v_1 + v_2 + v_3)$?
   (b) If $\beta' = \{v_1, v_2, v_1 + v_2 + v_3\}$, what is $[T]_{\beta'}^\gamma$? (You may assume that β' is a basis for V, but you should also know how to prove it)
3. Let $T : V \to V$ be a linear transformation. What is the relationship between $N(T)$ and $N(T^2)$? What is the relationship between $R(T)$ and $R(T^2)$?
4. Let V and W be vector spaces over a field F and let $\mathcal{L}(V, W)$ denote the set of linear transformations from V to W. How do we define addition and scalar multiplication in $\mathcal{L}(V, W)$? How would you show that $\mathcal{L}(V, W)$ is a vector space under these operations?

1. (a) i. W, ii. F^n, iii. $M_{m\times n}(F)$, iv. F^m

(b)

i. $[v]_\beta \in F^n$, which is not the domain of T.

ii. $v \in V$ but γ is a basis for W

iii. $[T]_\beta^\gamma$ is a matrix, and there is no general way to multiply a matrix by an element of a vector space

iv. $T(v)$ is an element of W, whereas $[S]_\beta^\gamma$ is only defined for S a linear transformation.

2. We have

$$[T(v_1+v_2+v_3)]_\gamma = [T]_\beta^\gamma [v_1+v_2+v_3]_\beta = [T]_\beta^\gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}.$$

Thus $T(v_1+v_2+v_3) = 6w_1 + 15w_2$.

3. If $T(x) = 0$, then $T^2(x) = T(T(x)) = T(0) = 0$, so if $x \in N(T)$ then $x \in N(T^2)$. That is, $N(T) \subseteq N(T^2)$.

If $y \in R(T^2)$, then there is some $x \in V$ with $T^2(x) = y$. That is, $T(T(x)) = y$. If $x' = T(x)$, then $T(x') = y$ so that $y \in R(T)$. Thus if $y \in R(T^2)$ then $y \in R(T)$, so that $R(T^2) \subseteq R(T)$.

4. If $T, S \in \mathcal{L}(V, W)$, then we define $T + S$ by $(T+S)(x) = T(x) + T(y)$. Similarly if $c \in F$ we define $(cT)(x) = cT(x)$. If one had already proven that for any set X, the set of functions from S to W, $\mathcal{F}(S, W)$, was an F-vector space with the same operations given, then you could prove that $\mathcal{L}(V, W)$ was a subspace of $\mathcal{F}(S, W)$. Otherwise, one would have to verify the vector space axioms.