Math 110, Fall 2012, Sections 109-110
Worksheet 5

1. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $\beta = \{(1,2),(-1,-3)\}$ is a basis for $\mathbb{R}^2$ for which $[T]_\beta = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Find $T(x,y)$.

3. (a) Find a nonzero $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 = 0$.

   (b) Show that there exists a non-zero linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T^2 = 0$.

   (c) If V is a finite-dimensional vector space, show that there is a non-zero linear transformation $S : V \to V$ such that $S^2 = 0$.

4. (a) Given a basis $\beta = \{x_1, \ldots, x_n\}$ for V, define the dual basis β^*.

   (b) Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. What is the dual basis β^*?

   (c) Let $\gamma = \{e_1, e_1 + e_2\}$. What is the dual basis γ^*?

5. Let $A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Let $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the left-multiplication linear transformation associated to A. Let β be the standard basis for $\mathbb{R}^2$, and let β^* be the dual basis for $(\mathbb{R}^2)^*$.

   (a) What are the domain and codomain of $(L_A)^t$? How is this different than L_{A^t}?

   (b) Compute $(L_A)^t$ on an arbitrary element of its domain.

   (c) Compute $[(L_A)^t]_{\beta^*}$, and comment.

1. Since AB is invertible, the linear transformation $L_{AB} : F^n \to F^n$ is invertible. (Recall that $L_{AB}(x) = ABx$.) By Theorem 2.15 again we have $L_{AB} = L_A L_B$. Since L_{AB} is invertible, $R(L_A L_B) = F^n$. But $F^n = R(L_A L_B) \subseteq R(L_A)$, so $R(L_A) = F^n$ as well. Hence, L_A is onto. By Theorem 2.5, L_A is invertible. We also have $N(L_B) \subseteq N(L_A L_B) = \{0\}$, so $N(L_B) = \{0\}$. Thus L_B is one-to-one, and therefore invertible by Theorem 2.5. Since L_A and L_B are invertible, so are A and B

However, if we don't assume that A and B are square matrices, it is possible for AB to be invertible but not have A and B invertible. For example, take

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is certainly invertible, but A and B can't be invertible because invertibility is only defined for *square* matrices.

2. (Also see Example 2.5.3) If β' is the standard basis for $\mathbb{R}^2$, then it will be easy to write down a formula for $T(x, y)$ once we know $[T]_{\beta'}$. If Q is the change-of-coordinates matrix from β to β', then $Q[T]_\beta Q^{-1} = [T]_{\beta'}$ by Theorem 2.23. Homework problem 2.5.2(a) tells us that

$$Q = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix},$$

so

$$[T]_{\beta'} = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}.$$

Since β' is the standard basis, T is just given by left-multiplication by $[T]_{\beta'}$. Thus

$$T(x, y) = \begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x - y \\ 6x - y \end{pmatrix}.$$

3. (a) Take $A = E_{1n}$. That is, A is the matrix with a 1 in the top right entry, and zero everywhere else. Then you can check that $A^2 = 0$:

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} = 0.$$

(b)Let $T = L_A$, where A is the matrix from part (a). Then by Theorem 2.15(e), we have

$$T^2 = (L_A)^2 = L_{A^2} = L_0 = 0.$$

(c) Suppose $\dim V = n$, and let $\phi : V \to F^n$ be an isomorphism. Such a ϕ exists by Theorem 2.19, or more specifically if we take any basis β for V, then $\phi(v) = [v]_\beta$ is such an isomorphism by Theorem 2.20. Then define

$$S = \phi^{-1} T \phi$$

where T is as in part (b). Then

$$S^2 = (\phi^{-1} T \phi)(\phi^{-1} T \phi) = \phi^{-1} T^2 \phi = \phi^{-1} 0 \phi = 0.$$

4. (a) $\beta^* = \{f_1, \ldots, f_n\}$ where $f_i(x_j) = \delta_{ij}$. See Theorem 2.24 and the following definition to see that this does indeed define a basis for V^*.
(b) By definition, $f_1(x, y) = f_1(xe_1 + ye_2) = x$ and $f_2(x, y) = f_2(xe_1 + ye_2) = y$ are the elements of $(\mathbb{R}^2)^*$ for which $\beta^* = \{f_1, f_2\}$. (c) Let $\gamma^* = \{f_1, f_2\}$ be the dual basis. By the definition of dual basis, we have

$$\begin{aligned} 1 &= f_1(1, 0) = f_1(e_1) \\ 0 &= f_1(1, 1) = f_1(e_1) + f_1(e_2). \end{aligned}$$

Thus we must have $f_1(e_2) = -1$ and so $f_1(x, y) = f_1(xe_1 + ye_2) = x - y$. Similarly,

$$\begin{aligned} 0 &= f_2(1, 0) = f_2(e_1) \\ 1 &= f_2(1, 1) = f_2(e_1) + f_2(e_2). \end{aligned}$$

Thus we must have $f_2(e_2) = 1$ and $f_2(x, y) = f_2(xe_1 + ye_2) = y$.

5. (a) $L_A : \mathbb{R}^2 \to \mathbb{R}^2$, so $(L_A)^t : (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$. However, $L_{A^t} : \mathbb{R}^2 \to \mathbb{R}^2$. (b) (Note that I've changed the numbers slightly from the worksheet handed out in class.) Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$, and let $\beta^* = \{f_1, f_2\}$ be its dual basis. Then

$$[(L_A)^t(f_1)](x, y) = f_1(L_A(x, y)) = f_1(A \cdot (x, y)) = f_1(x + 3y, x + 2y) = x + 3y,$$

and

$$[(L_A)^t(f_2)](x, y) = f_2(L_A(x, y)) = f_2(A \cdot (x, y)) = f_1(x + 3y, x + 2y) = x + 2y.$$

So let $g_1 = (L_A)^t(f_1)$, so that $g_1 \in (\mathbb{R}^2)^*$ and $g_1(x, y) = x + 3y$. Similarly, let $g_2 = (L_A)^t(f_2)$ so that $g_2(x, y) = x + 2y$. Then $(L_A)^t$ is given on an arbitrary element of its domain by

$$(L_A)^t(c_1 f_1 + c_2 f_2) = c_1 g_1 + c_2 g_2.$$

(c) Since $f_1(x, y) = x$ and $f_2(x, y) = y$, we have $g_1 = f_1 + 3f_2$ and $g_2 = f_1 + 2f_2$. Thus $[(L_A)^t(f_1)]_{\beta^*} = (1, 3)$ and $[(L_A)^t(f_2)]_{\beta^*} = (1, 2)$. The matrix $[(L_A)^t]_{\beta^*}$ has those coordinate vectors for columns, so

$$[(L_A)^t]_{\beta^*} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}.$$

Observe that

$$([L_A]_\beta)^t = [(L_A)^t]_{\beta^*},$$

where the transpose on the left is the matrix transpose, and the transpose on the right is the operator transpose. This illustrates Theorem 2.25.