Math 110, Fall 2012, Sections 109-110
Worksheet 6

1. Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$.
   (a) Write A^{-1} as a product of elementary matrices.
   (b) Write A as a product of elementary matrices
2. True or false? If true, provide proof. If false, provide a counterexample.
   (a) If A is an $m \times n$ matrix with a set of three linearly independent columns, then it also has a set of three linearly independent rows.
   (b) Elementary row operations preserve the rank of A.
   (c) Elementary column operations preserve the rank of A.
   (d) Elementary row operations on A preserve the range of L_A.
   (e) Elementary column operations on A preserve the range of L_A.
   (f) Every $n \times n$ matrix can be written as a product of elementary matrices.
3. For each of the following linear transformations, determine if T is invertible and compute T^{-1} if applicable.
   (a) $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p) = p'$.
   (b) $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined by $T(p) = (p(-1), p(0), p(1))$.
4. Suppose that D can be transformed into B using row and column operations. Prove that D' can be transformed into B' using row and column operations, where
$$D' = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & D \end{array}\right), \quad B' = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right).$$
5. Suppose that $A \in M_{m\times n}(F)$ and $b \in F^m$. Prove that if $\operatorname{rank}(A \mid b) = \operatorname{rank} A$, then there exists $x \in F^n$ such that $Ax = b$.

1. (a) Row reduction tells us
$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix},$$

and thus

$$\begin{pmatrix}1 & -4\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1/2\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 4\\ 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Thus A is invertible and

$$A^{-1} = \begin{pmatrix}1 & -4\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1/2\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

(b) We have

$$\begin{aligned}A &= \left(\begin{pmatrix}1 & -4\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1/2\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\right)^{-1}\\ &= \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 0 & -1/2\end{pmatrix}^{-1}\begin{pmatrix}1 & -4\\ 0 & 1\end{pmatrix}^{-1}\\ &= \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}.\end{aligned}$$

2. (a) True, since the rank of A is the maximal number of linearly independent columns and the maximal number of linearly independent rows.

(b) True, see book.

(c) True, see book.

(d) False. The ranges of L_A and $L_{A'}$ are distinct for

$$A = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad A' = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}.$$

(e) True. Performing elementary column operations results on A in a matrix AC, with C invertible. The claim is that $R(L_A) = R(L_{AC})$. If $y \in R(L_A)$, then there is an $x \in F^n$ with $Ax = y$. Then $AC(C^{-1}x) = y$, so $y \in R(L_{AC})$. Conversely, if $y \in R(L_{AC})$, there is some $x \in F^n$ with $ACx = y$. Then $A(Cx) = y$, so $y \in R(L_A)$. Thus we have shown that $R(L_{AC}) = R(L_A)$, as desired.

(f) False, since products of elementary matrices are invertible, and e.g. the zero matrix is not.

3. (a) Since $T(1) = 0$, we have $N(T) \neq \{0\}$ so T is not invertible. Alternatively, one could compute $[T]_\beta$ for some basis β of $P_3(\mathbb{R})$ (e.g. the standard basis) and see that this matrix is not invertible.

(b) If $\beta = \{1, x, x^2\}$ and $\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$, then we have

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

This matrix is invertible, with

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Thus we have

$$[T^{-1}(a,b,c)]_\beta = [T^{-1}]_\gamma^\beta[(a,b,c)]_\gamma = [T^{-1}]_\gamma^\beta \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2b \\ c-a \\ a-2b+c \end{pmatrix}.$$

Inverting the coordinates, we get

$$T^{-1}(a,b,c) = b + \frac{1}{2}(c-a)x + \frac{1}{2}(a-2b+c)x^2.$$

4. The idea is that we can just apply those same row operations that take D to B to the bottom part of the bigger matrices. More formally, if D can be transformed into B then there is an invertible matrix C such that $CD = B$. Then we have

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & C \end{array}\right) D' = B'.$$

Since the matrix multiplying D' on the left is invertible, we have that D' and B' are row-equivalent.

5. Let $C = (A \mid b)$. If $v_1, \dots, v_n \in F^m$ are the columns of A, then

$$\operatorname{Col} A = \operatorname{span}\{v_1, \dots, v_n\} \subseteq \operatorname{span}\{v_1, \dots, v_n, b\} = \operatorname{Col} C.$$

Here $\operatorname{Col} A$ is the column space of A (i.e. the span of the columns of A). However, $\operatorname{rank} A = \dim \operatorname{Col} A$ by definition, so $\dim \operatorname{Col} A = \dim \operatorname{Col} C$ by the hypothesis of the problem. Thus we must have $\operatorname{Col} A = \operatorname{Col} C$, and in particular $b \in \operatorname{Col} A$. Thus there are scalars $x_1, \dots, x_n$ such that

$$b = x_1 v_1 + \cdots + x_n v_n.$$

But then if $x = (x_1, \dots, x_n)$, we have $Ax = b$ by the definition of matrix multiplication.