Math 110, Fall 2012, Sections 109-110
Worksheet 7

1. What does it mean for two systems of equations to be *equivalent*? Give an example of two distinct but equivalent systems of linear equations.

2. (a) How do you find a basis for the column space of a matrix? Carefully justify why your method works, citing theorems where appropriate.
   (b) How do you find a basis for the null space of a matrix? Carefully justify why your method works, citing theorems where appropriate.
   (c) Apply your methods to find bases for the column space and the null space of

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ -2 & 4 & 10 & 8 \\ 1 & -2 & -5 & -4 \end{pmatrix}.$$

3. Are the following statements true or false? If true, justify your answer. If false, provide a counterexample.
   (a) If A is row equivalent to A', then $Ax = b$ is consistent if and only if $A'x = b$ is consistent.
   (b) The $n \times n$ matrix A is invertible if $Ax = 0$ has the trivial solution.

4. Prove that $Ax = b$ is consistent if and only if $\operatorname{rank} A = \operatorname{rank}(A \mid b)$.

5. Suppose $A \in M_{n\times n}(\mathbb{R})$ and $b \in \mathbb{R}^n$. Prove that if $Ax = b$ is consistent, then it either has one solution, or infinitely many solutions. For bonus points, use the words"homogeneous" in your response.

1. Two systems of equations are equivalent if they have the same solution sets. E.g. $x_1 + x_2 = 3$ and $2x_1 + 2x_2 = 6$ are equivalent.

2. (a) Reduce the matrix to reduced row echelon form, look at the leading 1 of each non-zero row. The columns that these 1's are in, in A, form a basis for $\operatorname{Col} A$. This follows from theorem 3.16(c), which says that a maximal linearly independent set of the columns of $\operatorname{rref}(A)$ will also be a maximal linearly independent subset of the columns of A.

(b) Let $B = \operatorname{rref}(A)$. The systems corresponding to $Ax = 0$ and $Bx = 0$ are equivalent (since row operations result in equivalent systems). Thus one finds a basis for the solution space of $Bx = 0$, as described in the book. It's worth noting that one can read the rank, and therefore the dimension of the null space, right from B , and it suffices to find a linearly independent or spanning set of the appropriate size.

3. (a) False. E.g. of the following, the first is consistent but the second is not:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) False. If $A = 0$, then $Ax = 0$ has the trivial solution $x = 0$. The statement becomes true by judiciously adding the word "only."

4. Let $C = (A \mid b)$. Exercise 5 on Worksheet 6 gives a proof that if $\operatorname{rank} A = \operatorname{rank} C$, then $Ax = b$ is consistent. We now prove the converse, so assume there is some $x \in F^n$ with $Ax = b$. If $v_1, \ldots, v_n$ are the columns of A, and $x = (x_1, \ldots, x_n)$ then

$$Ax = x_1 v_1 + \cdots + x_n v_n.$$

Thus $b = Ax \in \operatorname{Col} A$. Thus

$$\operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{v_1, \ldots, v_n, b\}$$

and $\operatorname{Col} A = \operatorname{Col} C$. In particular, these spaces have the same dimension so $\operatorname{rank} A = \operatorname{rank} C$.

5. Since $Ax = b$ is consistent, it has some solution x_0. Let K be the solution set (*not* a subspace unless $b = 0$) of $Ax = b$, and let K_H be the solution space (always a subspace) to the homogeneous equation $Ax = 0$. We know that $K = \{x_0 + x : x \in K_H\}$. Since $x_0 + x = x_0 + x'$ if and only if $x = x'$, this means that K and K_H have the same number of elements. But K_H is a vector space over an infinite field, so either it has one element (if $K_H = \{0\}$), or it has infinitely many elements (if $\dim K_H > 0$). Thus K has either one element, or infinitely many elements.