Math 110, Fall 2012, Sections 109-110
Worksheet 8

1. True or false? If true, justify. If false, provide a counterexample.

   (a) The determinant of an elementary matrix can be any element of the field.

   (b) A matrix $A \in M_{n\times n}(F)$ has rank n if and only if $\det A \neq 0$.

   (c) The determinant $\det : M_{n\times n}(F) \to F$ is a linear functional.

   (d) If $c \in F$, then $\det(cA) = c\det(A)$.

   (e) If you interchange two columns of a matrix, then the resulting matrix has determinant that is the opposite of the determinant of the original matrix.

   Solution:

   (a) False. Elementary matrices are invertible, so the determinant of an elementary matrix cannot be 0. Any other element of the field is possible (consider the elementary matrix that scales a row by a).

   (b) True. Both are equivalent to A being invertible.

   (c) False. The determinant is not additive and does not preserve scalar multiplication. E.g.

   $$\det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq 2 \cdot \det \mathrm{Id}_2 \,.$$

   (d) False. See (c).

   (e)True. Interchanging two columns is the same as multiplying on the right by an elementary matrix obtained by interchanging two columns (or rows) of the identity. Such elementary matrices have determinant -1.

2. A matrix $M \in M_{n\times n}(F)$ is called *nilpotent* if there exists a $k > 0$ for which $M^k = 0$. What can you say about the determinant of a nilpotent matrix?

   Solution: The determinant of a nilpotent matrix must be 0. We have

   $$0 = \det 0 = \det(M^k) = \det(M)^k.$$

   In a field if $a^k = 0$ then $a = 0$, so we must have $\det(M) = 0$.

3. Suppose $A, B \in M_{2013\times 2013}(\mathbb{R})$. Prove that we cannot have $AB = -BA$ with both A and B invertible. Bonus: construct invertible two by two matrices A and B with $AB = -BA$.

   Solution: The key observation is that $\det(-B) = (-1)^{2013}\det B$. So we have

   $$\det(A)\det(B) = (-1)^{2013}\det(B)\det(A) = -\det(A)\det(B).$$

   Thus $\det(A)\det(B) = 0$ so either A or B is not invertible. For 2×2 matrices, we can put

   $$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

4. Suppose that $D \in M_{n\times n}(F)$ is upper-triangular. In terms of the entries of D, what is the determinant of D?

   Solution: The product of the diagonal entries.

5. Suppose C is an $m \times m$ matrix. Calculate the determinant of the $(n+m)$ by $(n+m)$ matrix

   $$C' = \begin{pmatrix} C & B \\ 0 & I_n \end{pmatrix}.$$

   Solution: One can prove by induction that $\det C' = \det C$ (expand across the bottom row).