**Math 110, Fall 2012, Sections 109-110**
**Worksheet 9**

1. Give an example of a matrix that is:

   (a) Diagonalizable and invertible.

   (b) Not diagonalizable, but invertible.

   (c) Not invertible, but diagonalizable.

   (d) Neither invertible nor diagonalizable.

   **Solution:**

   $$\text{(a)}\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{(c)}\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{(d)}\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   To see that (b) and (c) are not diagonalizable, one could check directly or use the fact that if a matrix has a single eigenvalue, it is diagonalizable if and only if it is a scalar multiple of the identity. (To prove this, see what happens if $A = QDQ^{-1}$ and $A$ has only one eigenvalue).

2. Recall that a *nilpotent* linear operator $T : V \to V$ is one for which there exists a $k > 0$ with $T^k = 0$.

   (a) What can you say about the eigenvalues of a nilpotent linear operator?

   (b) What is the characteristic polynomial of a nilpotent linear operator (assume $F = \mathbb{C}$ for simplicity)?

   (c) When is a nilpotent linear operator diagonalizable?

   (d) Prove that if $T$ is nilpotent, then $I + T$ is invertible.

   **Solution:**

   (a) If $x$ is an eigenvector with eigenvalue $\lambda$ and $T^k = 0$, then $T^k x = \lambda^k x = 0$, so $\lambda^k = 0$. It follows that $\lambda = 0$, so the only possible eigenvalue of $T$ is 0.

   (b) The characteristic polynomial of $T$ splits as $(-1)^n(t - \lambda_1) \cdots (t - \lambda_k) = (-1^n)t^n$.

   (c) If $T$ were diagonalizable, there would exist a basis of eigenvectors $\beta$ such that $[T]_\beta = 0$. But then $T = 0$. So 0 is the only diagonalizable nilpotent operator.

   (d) $\det(I + T)$ is the characteristic polynomial of $T$, evaluated at $t = -1$. Thus $\det(I + T) = (-1)^n(-1)^n = 1$, and so $I + T$ is invertible.

3. Let

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 47 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}.$$

(a) What are the eigenvalues of $A$? What are the corresponding eigenvectors?

(b) Compute $([L_A]_\gamma)^3$ for whatever basis $\gamma$ you want to pick.

**Solution:**

(a) Note that

$$\begin{pmatrix} 4 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}^{-1}$$

The eigenvalues of $A$ are $\pi$ and 47, and the corresponding eigenvectors are, respectively, $c(1,3)^t$ and $d(1,4)^t$ with $c, d \neq 0$.

(b) I pick $\gamma = \{(1,3)^t, (1,4)^t\}$. Then $[L_A]_\gamma$ is diagonal and

$$[L_A]_\gamma^3 = \begin{pmatrix} \pi^3 & 0 \\ 0 & 47^3 \end{pmatrix}.$$

4. Suppose that $T$ is an operator on $V$ , and that $u$ and $v$ are eigenvectors for $T$. If $u+v$ is an eigenvector for $T$ with eigenvalue $\lambda$, what can you say about the eigenvalues of $u$ and $v$?

If $u$ and $v$ are linearly dependent, then $u$, $v$ and $u+v$ are all multiples of $u$, and thus have the same eigenvalue $\lambda$. So assume $u$ and $v$ are linearly independent.

Let $\alpha$ and $\beta$ be the eigenvalues of $u$ and $v$, respectively. We have

$$\lambda(u+v) = T(u+v) = T(u) + T(v) = \alpha u + \beta v.$$

Thus $(\lambda - \alpha)u + (\lambda - \beta)v = 0$. By linear independence, $\lambda = \alpha = \beta$.

5. Suppose $p(x) \in P_k(F)$ is given by

$$p(x) = a_k x^k + \cdots + a_1 x + a_0.$$

Recall that if $A \in M_{n\times n}(F)$, we can define

$$p(A) = a_k A^k + \cdots + a_1 A + a_0 I_n.$$

Prove that if $A$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$, then $p(A)$ is diagonalizable with eigenvalues $p(\lambda_1), \ldots, p(\lambda_n)$. (Hint: if $D$ is diagonal, what is $p(D)$?)

**Solution:** Since $A$ is diagonalizable, it can be written as $A = QDQ^{-1}$ where $D$ is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$. Then

$$\begin{aligned} p(A) &= a_k(QDQ^{-1})^k + \cdots + a_1 QDQ^{-1} + a_0 I_n \\ &= a_k QD^k Q^{-1} + \cdots + a_1 QDQ^{-1} + a_0 I_n \\ &= Q(a_k D^k + \cdots a_1 D + a_0 I_n)Q^{-1} \\ &= Qp(D)Q^{-1}. \end{aligned}$$

Observe that $p(D)$ is diagonal with diagonal entries $p(\lambda_1), \dots, p(\lambda_n)$, which gives the desired result.