Math 1B, Fall 2008
Section 107

Quiz 11 Solutions

(1) *Evaluate*

$$\int_0^2 z^2 \ln z \; dz.$$

Note that this integral is improper at $z = 0$ (well, it turns out that $z^2 \ln z$ is continuous on $[0, 2]$, but that's not obvious). Integrating by parts (with $u = \ln z$ and $dv = z^2 \; dz$) we get

$$\begin{aligned}
\int_0^2 z^2 \ln z \; dz &= \frac{1}{3} z^3 \ln z \Big]_0^2 - \frac{1}{3} \int_0^2 z^2 \; dz \\
&= \frac{1}{3} z^3 \ln z \Big]_0^2 - \frac{1}{9} z^3 \Big]_0^2 \\
&= \frac{8}{3} \ln 2 - \frac{8}{9} - \frac{1}{3} \lim_{z\to 0} z^3 \ln z.
\end{aligned}$$

Recognizing $\lim_{z\to 0} z^3 \ln z$ as an indeterminate form, we use L'Hopital to calculate

$$\lim_{z\to 0} z^3 \ln z = \lim_{z\to 0} \frac{\ln z}{z^{-3}} = \lim_{z\to 0} \frac{z^{-1}}{-3z^{-4}} = -\lim_{z\to 0} \frac{z^3}{3} = 0.$$

So the answer is

$$\int_0^2 z^2 \ln z \; dz = \frac{8}{3} \ln 2 - \frac{8}{9}.$$

(2) *Solve the initial value problem*

$$y'' = -3y, \qquad y(0) = 1, \qquad y'(0) = 3.$$

The differential equation in question can be rewritten $y'' + 3y = 0$, so the characteristic polynomial is $r^2 + 3 = 0$, which has roots $\pm i\sqrt{3}$. So we are in case 3, with $\alpha = 0$ and $\beta = \sqrt{3}$. So our solution must be of the form

$$y(x) = C_1 \cos\left(\sqrt{3}x\right) + C_2 \sin\left(\sqrt{3}x\right).$$

We now use the initial conditions to find C_1 and C_2. Our initial conditions say

$$\begin{aligned}
1 = y(0) &= C_1 \cos 0 + C_2 \sin 0 = C_1, \\
3 = y''(0) &= -\sqrt{3}C_1 \sin 0 + \sqrt{3}C_2 \cos 0 = \sqrt{3}C_2.
\end{aligned}$$

So $C_1 = 1$ and $C_2 = \sqrt{3}$. This yields the solution

$$y(x) = \cos\left(\sqrt{3}x\right) + \sqrt{3}\sin\left(\sqrt{3}x\right).$$

(3) *Solve the initial value problem for y (as a function of x) for $0 < x < \pi/2$:*

$$y' \sin x = (a + y)\cos x, \qquad y(\pi/3) = a,$$

where a is some constant.

We begin by separating variables to get

$$\frac{dy}{a+y} = \frac{\cos x}{\sin x}\ dx.$$

Next, integrate both sides:

$$\int \frac{dy}{a+y} = \ln|a+y| + C,$$

and (using the subtitution $u = \sin x$)

$$\int \frac{\cos x}{\sin x}\ dx = \int u^{-1}\ du = \ln|\sin x| + C.$$

So we have

$$\ln|a+y| = \ln|\sin x| + C_1$$

and consequently

$$|a+y| = e^{C_1}|\sin x|.$$

Letting $C_2 = \pm e^{C_1}$ (choosing sign as necessary), and rearranging we get

$$y = C_2 \sin x - a.$$

To find C_2, subtitute $y(\pi/3) = a$ to get $a = C_2\frac{\sqrt{3}}{2} - a$, or $C_1 = \frac{4a}{\sqrt{3}}$. Thus our final answer is

$$y = \frac{4a}{\sqrt{3}}\sin x - a.$$