Math 1B, Fall 2008
Section 108

Quiz 12 Solutions

(1) *Solve the initial value problem*

$$\begin{cases} y'' + y' - 2y = x + \sin 3x \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$$

NB: This problem involved a little too much algebra/arithmatic for the time allowed. The complementary equation $y'' + y' - 2y = 0$ has general solution $C_1e^{-2x} + C_2e^x$. To solve the non-homogenous equation, we use the principle of superposition and find two functions y_{p_1} and y_{p_2} satisfying

$$\begin{aligned} y''_{p_1} + y'_{p_1} - 2y_{p_1} &= x \\ y''_{p_2} + y'_{p_2} - 2y_{p_2} &= \sin 3x. \end{aligned}$$

(Then $y_p = y_{p_1} + y_{p_2}$). We guess $y_{p_1} = Ax + B$ (because x is a first-degree polynomial). Plugging it in gives

$$\begin{aligned} x &= y''_{p_1} + y'_{p_1} - 2y_{p_1} \\ &= 0 + A - 2Ax - 2B. \end{aligned}$$

So we have $x = -2Ax + (A - 2B)$. Equating the coefficients, we get $1 = -2A$, so $A = -\frac{1}{2}$. Also, $0 = A - 2B = -\frac{1}{2} - 2B$, so $B = -\frac{1}{4}$. So $y_{p_1} = -\frac{1}{2}x - \frac{1}{4}$.

Now we guess $y_{p_2} = C \sin 3x + D \cos 3x$, and this gives

$$\begin{aligned} y'_{p_2} &= 3C \cos 3x - 3D \sin 3x \\ y''_{p_2} &= -9C \sin 3x - 9D \cos 3x. \end{aligned}$$

Now plugging into the equation for y_{p_2} gives

$$\begin{aligned} \sin 3x &= y''_{p_2} + y'_{p_2} - 2y_{p_2} \\ &= -9C \sin 3x - 9D \cos 3x + 3C \cos 3x - 3D \sin 3x - 2(C \sin 3x + D \cos 3x) \\ &= -(11C + 3D) \sin 3x + (C - 9D) \cos x. \end{aligned}$$

Equating coefficients gives

$$\begin{aligned} 1 &= -11C - 3D \\ 0 &= 3C - 11D. \end{aligned}$$

Multiplying the second equation by $11/3$ and adding to the first gives $1 = -(130/3)D$, or $D = -3/130$. Substituting this into the first equation, we get $1 = -(130/11)C$, or $C = -11/130$. So $y_{p_2} = -(11/130)\sin 3x - (3/130)\cos 3x$, and the general solution is

$$-\frac{1}{2}x - \frac{1}{4} - \frac{11}{130}\sin 3x - \frac{3}{130}\cos 3x + C_1 e^{-2x} + C_2 e^x.$$

Plugging in $y(0) = 1$ and $y'(0) = 0$ gives

$$\begin{aligned} 1 &= -\frac{1}{4} - \frac{3}{130} + C_1 + C_2 \\ 0 &= -\frac{1}{2} - \frac{33}{130} - 2C_1 + C_2. \end{aligned}$$

Doubling the first and adding it to the second gives $2 = -1 - (39/130) + 3C_2$, or $C_2 = 143/130 = 11/10$. Now plugging into the first equation gives $C_1 = 9/52$. Thus the answer to the initial value problem is

$$-\frac{1}{2}x - \frac{1}{4} - \frac{11}{130}\sin 3x - \frac{3}{130}\cos 3x + \frac{9}{52}e^{-2x} + \frac{11}{10}e^x.$$

Phew.

(2) *Find the general solution of*

$$y'' - 4y' - 5y = e^{-1}.$$

We first solve the homogenous equation $y'' - 4y' - 5y = 0$. The characteristic polynomial is $x^2 - 4y - 5$, which has roots -1 and 5. So the general solution to the complementary equation is $C_1 e^{-x} + C_2 e^{5x}$. The right-hand side of our original equation satisfies the homogenous equation, so we guess a solution of the form $y_p = Axe^{-x}$. Differentiating yields

$$\begin{aligned} y_p' &= Ae^{-x} - Axe^{-x} \\ y_p'' &= Axe^{-x} - 2Ae^{-x}. \end{aligned}$$

So we get

$$\begin{aligned} e^{-x} &= y_p'' - 4y_p' - 5y_p \\ &= Axe^{-x} - 2Ae^{-x} - 4(Ae^{-x} - Axe^{-x}) - 5Axe^{-x} \\ &= -6Ae^{-x}. \end{aligned}$$

Thus $A = -1/6$. The general solution is therefore

$$-\frac{1}{6}xe^{-x} + C_1 e^{-x} + C_2 e^{5x}.$$

(3) *Suppose that* $a, b, c > 0$, *that* $b^2 = 4ac$, *and that* y *solves* $ay'' + by' + cy = 0$. *Find* $\lim_{x\to\infty} y(x)$.

The general solution of $ay''+by'+cy = 0$ is $C_1e^{rx}+C_2xe^{rx}$, where $r = -b/2a$. Since y solves this equation, it must be of the indicated form, so there are constants C_1,C_2 such that $y = C_1e^{rx} + C_2xe^{rx}$. Now compute

$$\begin{aligned}
\lim_{x\to\infty} y(x) &= \lim_{x\to\infty} C_1e^{rx} + C_2xe^{rx} \\
&= \lim_{x\to\infty} \frac{C_1}{e^{bx/2a}} + \frac{C_2x}{e^{bx/2a}} \\
&= 0 + \lim_{x\to\infty} \frac{C_2x}{e^{bx/2a}} \quad \text{(because } a, b > 0\text{)} \\
&= 0 + \lim_{x\to\infty} \frac{2aC_2}{be^{bx/2a}} \qquad \text{(L'Hopital)} \\
&= 0.
\end{aligned}$$