Math 1B, Fall 2008
Section 108

Quiz 13 Solutions

(1) *Solve the boundary value problem*

$$y'' + 2\frac{y'}{x} = 0, \qquad y(1) = 15, \qquad y(3) = 25.$$

Let's make the subtitution $w = y'$. This yields the differential equation

$$w' + 2\frac{w}{x} = 0,$$

or in Leibniz notation:

$$\frac{dw}{dx} + 2\frac{w}{x} = 0.$$

Rearranging and separating variables, we get

$$\int \frac{dw}{w} = -2\int \frac{dx}{x},$$

which gives $\ln|w| = -2\ln|x| + C_1$. Exponentiating gives

$$|w| = e^{-2\ln|x|}e^{C_1} = (e^{\ln|x|})^{-2}e^{C_1} = |x|^{-2}\, e^{C_1}.$$

The usual trick now gives us $w = C_2x^{-2}$. Since $w = y'$, we get

$$y = \int w\ dx = \int C_2x^{-2} = -C_2x^{-1} + C_3.$$

Plugging in $y(1) = 15$ and $y(3) = 25$, we get

$$\begin{aligned} 15 = y(1) &= -C_2 + C_3 \\ 25 = y(3) &= -\frac{C_2}{3} + C_3. \end{aligned}$$

Subtracting the first from the second, we get $10 = \frac{2}{3}C_2$, or $C_2 = 15$. Plugging into the first equation gives $C_3 = 30$. Thus our solution is

$$y = -\frac{15}{x} + 30.$$

(2) *Suppose the coefficient* c_n *of a power series satisfy*

$$c_n + (n+2)c_{n+2} = 0$$

for all $n \geq 0$. *Which coefficients can be chosen arbitrarily? Express the other coefficients in terms of these.*

Expressing the higher coefficient in terms of the lower one, we get

$$c_{n+2} = -\frac{c_n}{n+2}.$$

Now, let's write out the first few equations.

$$\begin{array}{ll}
n = 0 & c_2 = -\dfrac{c_0}{2} \\[2ex]
n = 1 & c_3 = -\dfrac{c_1}{3} \\[2ex]
n = 2 & c_4 = -\dfrac{c_2}{4} = \dfrac{c_0}{2 \cdot 4} \\[2ex]
n = 3 & c_5 = -\dfrac{c_3}{5} = \dfrac{c_1}{3 \cdot 5} \\[2ex]
n = 4 & c_6 = -\dfrac{c_4}{6} = -\dfrac{c_0}{2 \cdot 4 \cdot 6} \\[2ex]
n = 5 & c_7 = -\dfrac{c_5}{7} = -\dfrac{c_1}{3 \cdot 5 \cdot 7}.
\end{array}$$

We see that c_0 and c_1 do not depend on any previous coefficients, and can therefore be chosen arbitrarily. In terms of these two, we have separate formulas for even and odd coefficients:

$$\begin{aligned}
c_{2n} &= \frac{(-1)^n c_0}{2 \cdot 4 \cdots 2n} \\
c_{2n+1} &= \frac{(-1)^n c_1}{3 \cdot 5 \cdots (2n+1)}
\end{aligned}$$

when $n \geq 1$. Possible (but probably unnecessary) simplifications are

$$\begin{aligned}
c_{2n} &= \frac{(-1)^n c_0}{2 \cdot 4 \cdots 2n} = \frac{(-1)^n c_0}{2^n n!}, \\
c_{2n+1} &= \frac{(-1)^n c_1}{3 \cdot 5 \cdots (2n+1)} = \frac{(-1)^n (2n+1)! c_1}{2 \cdot 4 \cdots 2n} = \frac{(-1)^n (2n+1)! c_1}{2^n n!}.
\end{aligned}$$

(3) *A spring, with rest length* $0.75m$ *and a* $5kg$ *mass attached, requires* $50N$ *of force to hold it stretched to a length of* $1m$. *If it is then released from this position, where will the mass be after* t *seconds?*

When it is stretched to a length of $1m$, the spring is stretched $0.25m$ beyond rest length. So we have the equation $50 = 0.25k$ or $k = 200N/m$. Thus our differential equation is

$$0 = mx'' + kx = 5x'' + 200x.$$

This has general solution $C_1 \cos \sqrt{40}t + C_2 \sin \sqrt{40}t$. What initial conditions do we know about the spring? It is released (from rest), and therefore we have $x'(0) = 0$. It is released from being stretched $0.25m$, so $x(0) = 0.25$. Plugging in we get $C_1 = 0.25$ and $C_2 = 0$, so our result is

$$x(t) = 0.25 \cos(\sqrt{40}t) = 0.25 \cos(2\sqrt{10}t).$$