Math 1B, Fall 2008
Section 108

Quiz 7 Solutions

(1) *Determine whether the following series is absolutely convergent, conditionally convergent, or divergent,*

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}.$$

We test first for absolute convergence with the Integral Test. In this case, $|a_n| = \frac{1}{n \ln n}$. Because x and $\ln x$ are increasing, $f(x) = \frac{1}{x \ln x}$ is decreasing. The function f is also continuous and positive when $x \geq 2$, so the Integral Test applies. Integrating with $u = \ln x$ gives

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du = \lim_{b \to \infty} \ln b - \ln(\ln 2),$$

which diverges. So $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges as well. However, because $\frac{1}{n \ln n}$ is decreasing (from before) and converges to 0, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the Alternating Series Test. We conclude that the series is conditionally convergent.

(2) *Find the radius and interval of convergence for*

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}.$$

We apply the ratio test with $a_n = \frac{(-2)^n x^n}{\sqrt[4]{n}}$:

$$\begin{aligned}
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \to \infty} \left| \frac{(-2)^{n+1} x^{n+1}}{\sqrt[4]{n+1}} \cdot \frac{\sqrt[4]{n}}{(-2)^n x^n} \right| \\
&= \lim_{n \to \infty} \sqrt[4]{\frac{n}{n+1}} \cdot 2\,|x| \\
&= 2\,|x|.
\end{aligned}$$

By the Ratio Test, the series converges when $|x| < \frac{1}{2}$ and diverges when $|x| > \frac{1}{2}$, and the radius of convergence must be $R = \frac{1}{2}$. To find the interval of convergence, we test the endpoints:

$$x = \frac{1}{2}: \qquad \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}} \quad \text{converges by Alternating Series Test.}$$

$$x = -\frac{1}{2}: \qquad \sum_{n=2}^{\infty} \frac{1}{\sqrt[4]{n}} \quad \text{diverges by } p\text{-test } (p = \tfrac{1}{4} \leq 1).$$

So the interval of convergence is $(-\frac{1}{2}, \frac{1}{2}]$.

(3) *Find a power series expansion and its interval of convergence for*

$$f(x) = \frac{x}{9-x}.$$

Trying to get something of the form $\frac{1}{1-*}$, we calculate

$$\begin{aligned}
\frac{x}{9-x} &= \frac{x}{9} \cdot \frac{1}{1-\frac{x}{9}} \\
&= \frac{x}{9} \sum_{n=0}^{\infty} \left(\frac{x}{9}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{x^{n+1}}{9^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{x^n}{9^n}.
\end{aligned}$$

When we substituted $\frac{1}{1-\frac{x}{9}} = \sum_{n=0}^{\infty} \left(\frac{x}{9}\right)^n$, the sum in question converges when $\frac{|x|}{9} < 1$ and diverges when $\frac{|x|}{9} \geq 1$ by the rules for geometric series. Multiplying the series by $\frac{x}{9}$ won't change the interval of convergence, so the interval of convergence is $-9 < x < 9$, or $(-9, 9)$.