Math 32, Spring 2010, Section 101
Worksheet 12 Solutions

Work through the following problems in groups of about four. Take turns writing; everyone should get a chance to write for some of the problems. It's more important to understand the problems than to do all of them.

1. Find all solutions to the following equations

   (a) $\sin\theta + \dfrac{1}{\sqrt{2}} = 0$

   (b) $\cos\theta + 2\sec\theta = -3$ (hint: multiply through by cosine)

   (c) $\tan 2\theta = -1$

   (d) $\sin\frac{\theta}{2} = \frac{1}{2}$

   (a) Rearranging turns it into $\sin\theta = -\frac{1}{\sqrt{2}}$. From our knowledge of the unit circle, the solutions to this equation in $[0, 2\pi)$ are $5\pi/4$ and $7\pi/4$. So the solutions to the equation are $\theta = 5\pi/4 + 2\pi k$ or $\theta = 7\pi/4 + 2\pi k$ for every integer k.

   (b) Subtituting $\sec\theta = 1/\cos\theta$, we get $\cos\theta + \frac{2}{\cos\theta} = -3$. Multiplying through by cosine gives $\cos^2\theta + 2 = -3\cos\theta$ or $\cos^2\theta + 3\cos\theta + 2 = 0$. Factoring gives $(\cos\theta+2)(\cos\theta+1) = 0$. So $\cos\theta = -2$ or $\cos\theta = -1$. The first equation is impossible becuase $-1 \leq \cos\theta \leq 1$ for all θ. So our solution is all θ such that $\cos\theta = -1$. The only solution to this equation in $[0, 2\pi)$ is $\theta = \pi$, so all of the solutions are $\pi + 2\pi k = (2k+1)\pi$ for k an integer.

   (c) Let $\psi = 2\theta$. We are solving $\tan\psi = -1$. With tangent, we only need to find the solutions between 0 and π and add a multiple of π. In this case, from our knowledge of the unit circle the only solution between 0 and π is $3\pi/4$. Thus we get $\psi = 3\pi/4 + \pi k$ for an integer k. Since $\theta = \psi/2$, we get $\theta = \frac{1}{2}(3\pi/4 + \pi k)$ or $\theta = 3\pi/8 + (\pi/2)k$ for integers k.

   (d) Proceeding as in (c), let $\psi = \theta/2$. We are now solving $\sin\psi = \frac{1}{2}$. From our knowledge of the unity circle, the solutions to this for ψ in $[0, 2\pi)$ are $\pi/6$ and $5\pi/6$. Thus we get that all of the solutions are $\psi = \pi/6 + 2\pi k$ and $\psi = 5\pi/6 + 2\pi k$ for integers k. Since $\theta = 2\psi$, we get $\theta = 2\pi/6 + 4\pi k$ and $\theta = 10\pi/6 + 4\pi k$ for integer k. Simplifying, we get $\theta = \pi/3 + 4\pi k$ and $\theta = 5\pi/3 + 4\pi k$.

2. Let $z = 2 + 3i$ and $w = 2 + i$. Compute and simplify the following.

   (a) $z\overline{z}$ (b) w/z (c) $z - w$

   (a) $z\overline{z} = (2+3i)(2-3i) = 4 + 6i - 6i - 9i^2 = 4 - (-9) = 13$. A fact that can shorten this is that for any $s = a + ib$, we get $s\overline{s} = a^2 + b^2$.

   (b) $w/z = w\overline{z}/(z\overline{z}) = \dfrac{(2+i)(2-3i)}{13} = \dfrac{7-4i}{13} = \frac{7}{13} - \frac{4}{13}i$.

   (c) $(2+3i) - (2+i) = 2i$.

3. Use polynomial long division to find the quotients and the remainders.

   (a) $\dfrac{x^3 - 4x^2 + x - 2}{x - 5}$

   (b) $\dfrac{z^5 - 1}{z - 1}$

   (c) $\dfrac{4y^4 - y^3 + 2y - 1}{2y^2 - 3y - 4}$

   (a)

$$
\begin{array}{r|rrrr}
 & & x^2 & +x & +6 \\ \hline
x-5 & x^3 & -4x^2 & +x & -2 \\
 & -x^3 & +5x^2 & & \\ \cline{2-3}
 & & x^2 & +x & \\
 & & -x^2 & +5x & \\ \cline{3-4}
 & & & 6x & -2 \\
 & & & -6x & +30 \\ \cline{4-5}
 & & & & 28
\end{array}
$$

(b)

$$
\begin{array}{r}
z^4 + z^3 + z^2 + z + 1 \\
z - 1 \overline{\big) \; z^5 \qquad\qquad\qquad\qquad - 1} \\
\underline{- z^5 + z^4} \qquad\qquad\qquad \\
z^4 \qquad\qquad\qquad \\
\underline{- z^4 + z^3} \qquad\qquad \\
z^3 \qquad\qquad \\
\underline{- z^3 + z^2} \qquad \\
z^2 \qquad \\
\underline{- z^2 + z} \; \\
z - 1 \\
\underline{- z + 1} \\
0
\end{array}
$$

(c)

$$
\begin{array}{r}
2y^2 \; + \tfrac{5}{2}y + \tfrac{31}{4} \\
2y^2 - 3y - 4 \overline{\big) \; 4y^4 \; - y^3 \qquad\qquad + 2y \; - 1} \\
\underline{- 4y^4 + 6y^3 \; + 8y^2} \qquad\qquad\quad \\
5y^3 \; + 8y^2 \; + 2y \qquad \\
\underline{- 5y^3 + \tfrac{15}{2}y^2 + 10y} \qquad \\
\tfrac{31}{2}y^2 + 12y \; - 1 \\
\underline{- \tfrac{31}{2}y^2 + \tfrac{93}{4}y + 31} \\
\tfrac{141}{4}y + 30
\end{array}
$$

4. Which of the following are guaranteed to have a (potentially complex) solution by the fundamental theorem of algebra?

   (a) $\sqrt{3}x^{17} + \sqrt{2}x^{13} + \sqrt{5} = 0$

   (b) $17x^{\sqrt{3}} + 13x^{\sqrt{2}} + 1 = 0$

   (c) $\dfrac{1}{x^2 + 1} = 0$

   Only (a). The second one fails because the powers aren't whole numbers, and the last one fails because of the division.