Math 32, Spring 2010, Section 101
Worksheet 5

Work through the following problems in groups of about four. Take turns writing; everyone should get a chance to write for some of the problems. It's more important to understand the problems than to do all of them.

1. Sketch a graph of the curve $y = x^2 - 4x + 5$. Be sure to label the vertex, the y-intercept and any x-intercept(s).

   Completing the square, we get $x^2 - 4x + 5 = (x^2 - 4x + 4) - 4 + 5 = (x - 2)^2 + 1$. Thus the graph of the function is the parabola $y = x^2$, but translated 2 to the right, and one up. Looking at the graph, we can see that the vertex is $(2, 1)$ and that there are no x-intercepts. Plugging 0 into the equation, we get that the y-intercept is $(0, 5)$.

2. A 10in piece of wire is cut into two pieces of length x and y. These pieces of wire are bent into squares. Express the combined total area of the squares as a function of x.

   Let r and s be the lengths of the sides of the two squares. Our target equation is then $A = r^2 + s^2$. Our constraint equations are that $x = 4r$ (since x is the perimeter of the first square), $y = 4s$ (since y is the perimeter of the second square), and $x + y = 10$ (since the total amount of wire used was 10in.). We need to eliminate r and s in the target equation, so we solve the first two constraints to get $r = x/4$ and $s = y/4$. This yields $A = x^2/16 + y^2/16$. Solving the third constraint for y (since we are asked to give an answer as a function of x), we get $y = 10 - x$. This gives

   $$A = \frac{x^2}{16} + \frac{(10-x)^2}{16} = \frac{x^2}{16} + \frac{x^2 - 20x + 100}{16} = \frac{1}{16}(2x^2 - 20x + 100)$$

   .

3. Find the point of on the curve $y = \sqrt{x}$ that is nearest to the point $(3, 0)$. What is this minimum distance?

   Our target equation is the distance between the point (x, y) and $(3, 0)$. That is, the target equation is $d = \sqrt{(x-3)^2 + (y-0)^2} = \sqrt{(x-3)^2 + y^2}$. Our constraint equation is that $y = \sqrt{x}$. Subtituting this in, we get $d = \sqrt{(x-3)^2 + x} = \sqrt{x^2 - 5x + 9}$.

To minimize a square root, it is enough to minimize the expression inside the square root. So the minimum will occur for the same x that gives the minimum of $x^2 - 5x + 9$. Using the vertex formula, this is at $x = \frac{5}{2}$. So the x-coordinate of the closest point on the curve to $(3, 0)$ is $\frac{5}{2}$. What is the y-coordiante? That is given by the curve, $y = \sqrt{x} = \sqrt{5/2}$. Thus the closest point on the curve is $(5/2, \sqrt{5/2})$.

What is the distance at the closest point? We just need to know the distance between $(5/2, \sqrt{5/2})$ and $(3, 0)$, which can be obtained by plugging $x = 5/2$ into our formula for d above. That gives the minimum distance is

$$\sqrt{\left(\frac{5}{2}\right)^2 - 5 \cdot \frac{5}{2} + 9} = \sqrt{-25/4 + 9} = \sqrt{11/4} = \frac{\sqrt{11}}{2}$$

4. Graph the function $y = x^2(x-1)(x-3)^2$ by (a) finding the x- and y-intercepts, (b) marking the excluded regions, and (c) drawing a curve that fits this data.

The x-intercepts are the points where the y-value is 0. That is, $(0, 0)$, $(1, 0)$ and $(3, 0)$ (which were easy to find because the polynoimal was factored). The y-intercept is the point where $x = 0$, and the y-coordinate is found by plugging in $x = 0$. In this case, it's $(0, 0)$, which we already knew the graph went through. We now make a table, indexed by the intervals between the key numbers 0, 1, and 3, telling us the sign of the polynomial.

	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
$x^2(x-1)(x-3)^2$	$-$	$-$	$+$	$+$

So we can exclude the regions above $(-\infty, 0)$ and above $(0, 1)$. We can also exclude the regions below $(1, 3)$ and $(3, \infty)$. Using this information, we draw a graph as best we can.