Math 54, Spring 2009, Sections 109 and 112
Midterm 1 Review

Note: this sheet is not exhaustive. It's just a collection of some things that seemed important to me as I was writing it. It is not a substitute for studying.

Important terms

The following definitions are important to know. It's not enough to be able to recite definitions, you have to understand how they're used, and how they fit together (i.e. a subspace cannot have pivots, a matrix cannot be linearly independent, a matrix doesn't have a solution, etc.) and to be comfortable using them.

- Basics: <u>echelon form</u>, <u>reduced echelon form</u>, <u>pivots</u>, <u>augmented matrix</u>, <u>free variable</u>, <u>basic variable</u> (see sections 1.1 and 1.2)
- A *square* matrix A is called <u>invertible</u> if there is another matrix B of the same size such that $AB = BA = I_n$.
- An <u>elementary matrix</u> is a matrix that can be obtained from the identity matrix with a single elementary row operation.
- A collection of vectors $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ is called <u>linearly independent</u> if you cannot have $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ unless $c_1 = c_2 = \cdots = c_p = 0$. This is equivalent to $A\vec{x} = \vec{0}$ having only the trivial solution $\vec{x} = 0$, where $A = [\vec{v}_1|\cdots|\vec{v}_p]$. (Why are these equivalent?)
- Alternatively, $\vec{v}_1, \ldots, \vec{v}_p$ is <u>linearly dependent</u> if there are some $c_1, \ldots, c_p$, not all 0, such that $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$. This is equivalent to $A\vec{x} = \vec{0}$ having a non-trivial solution, where A is as above.
- A <u>subspace</u> H of $\mathbb{R}^n$ is a set of vectors that contains $\vec{0}$, and is closed under addition and scalar multiplicatoin.
- Given vectors $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, the <u>span</u> of these vectors is the set $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$, the set of all linear combinations of the given vectors. This is the smallest subspace of $\mathbb{R}^n$ containing those vectors.
- If $H = \text{Span}\,\vec{v}_1, \ldots \vec{v}_p$, then we say that $\{\vec{v}_1, \ldots, \vec{v}_p\}$ <u>spans</u> H, and that $\{\vec{v}_1, \ldots, \vec{v}_p\}$ <u>generate</u> H.

- A basis for a subspace H is a set of vectors in H which is both linearly independent and spans H. The dimension of H is the number of vectors in a (every) basis.

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it respects addition and scalar multiplication. That is, for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$ we have $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(c\vec{x}) = cT(\vec{x})$.

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called one-to-one if for every $y \in \mathbb{R}^m$, there is at most one $x \in \mathbb{R}^n$ satisfying $T(\vec{x}) = \vec{y}$. That is, no two vectors in $\mathbb{R}^n$ get sent to the same place.

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called onto if for every $y \in \mathbb{R}^m$, there is at least one $x \in \mathbb{R}^n$ satisfying $T(\vec{x}) = \vec{y}$. That is, everything in $\mathbb{R}^m$ gets hit by something.

- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the domain of T is $\mathbb{R}^n$, and the codomain of T is $\mathbb{R}^m$. The range of T is the subspace of $\mathbb{R}^m$ consisting of all things that are "hit" by T. How can you characterize onto linear transformations in terms of these words?

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called invertible if there is some $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(S(\vec{x})) = S(T(\vec{x})) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^n$. Note how this parallels the definition of invertible matrix.

- The standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the unique $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for every $x \in \mathbb{R}^n$. The columns of the standard matrix are $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, where e_j is the j-th column of I_n.

- The column space of a matrix is the span of its columns. The rank of that matrix is the dimension of the column space.

- The null space of a matrix (or linear transformation) is the set of all $\vec{x}$ such that $A\vec{x} = \vec{0}$ (or $T(\vec{x}) = \vec{0}$).

Be able to: row reduce a matrix, multiply two matrices (and know if the multiplication is defined), determine if a matrix is invertible and find it's inverse (if applicable), find the rank of a matrix, find bases for the null and column spaces of a matrix, find the standard matrix of a linear transformation, understand the relationship between matrix equations and vector equations and translate between the two, determine if a set is linearly independent, find determinants, understand the theorems (invertible matrix theorem, basis theorem, rank theorem, theorem 4 on p.43, just to name a few).