Math 54, Spring 2009, Sections 109 and 112
Midterm 1 Review Exercises
SOLUTIONS

These exercises don't cover some of the **very important** computational-type problems, including many of the things listed under the "be able to" section of the review sheet. You can find examples of those types of problems on the sample exam and in the sections of the book (including the supplemental exercises at the end of each chapter). These are a little more theoretical, and are aimed at making sure you have a good grasp of the ideas underlying the algorithms.

1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and A be its standard matrix. Complete the following table:

Property of T	Columns of A	Pivots of A	$A\vec{x} = \vec{b}$?
One-to-one	Lin. ind.	Every column	≤ 1 solution for every $\vec{b}$
Onto	Span $\mathbb{R}^m$	Every row	≥ 1 solution for every $\vec{b}$
Invertible	Basis for $\mathbb{R}^m$ $(m = n)$	Every row and column	exactly 1 solution for every $\vec{b}$

2) Let A be a 17×17 matrix such that $A^{12} = I_{17}$. What can you say about Rank(A)? Nul(A)? Find a basis for Col(A).

We have $AA^{11} = A^{11}A = I_{17}$, so A is invertible with $A^{-1} = A^{11}$. The columns of A therefore form a basis for $\mathbb{R}^{17}$ and thus Rank$(A) = 17$. By the Rank Theorem, this means $\dim \mathrm{Nul}(A) = 0$ and thus $\mathrm{Nul}(A) = \{\vec{0}\}$. As stated above, $\mathrm{Col}(A) = \mathbb{R}^{17}$, and we can pick any basis. The easiest choice is the standard basis $\{\vec{e}_1, \ldots, \vec{e}_{17}\}$.

3) (#10, p.184) Suppose A is invertible. Explain why A^TA is also invertible, and then show that $A^{-1} = (A^TA)^{-1}A^T$.

The easiest way to see that A^TA is invertible is to notice that $\det(A^TA) = \det(A)^2$, so if $\det(A) \neq 0$, then $\det(A^TA) \neq 0$ also. Now we can check that $((A^TA)^{-1}A^T)A = (A^TA)^{-1}(A^TA) = I$, so $(A^TA)^{-1}A^T$ is a left inverse for A. But A is invertible, so the left inverse must be its inverse (Hint: simplify $(A^TA)^{-1}A^TAA^{-1}$ in two different ways).

4) Suppose you have a square matrix such that $A^3 = 0$ (the zero matrix). Use matrix algebra to compute $(I - A)(I + A + A^2)$. Generalize to show that if $A^k = 0$ for some $k \geq 1$, then $(I - A)$ is invertible.

Expanding we get

$$(I - A)(I + A + A^2) = I + A + A^2 - (A + A^2 - A^3) = I - A^3 = I.$$

More generally, if $A^k = 0$ then

$$(I - A)(I + A + \cdots + A^{k-1}) = (I + A + \cdots + A^{k-1})(I - A) = I,$$

so $(I - A)$ is invertible.

5) True or False? If true, justify. If false, provide a counterexample. (Some of these are from p.102.)

(a) If $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$, so is $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$.

(b) If an $m \times n$ matrix A has a pivot in every column or has a pivot in every row, then it is invertible.

(c) If T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

(d) If A is a square matrix, then it can be written as a product of elementary matrices.

(e) If A is an $n \times n$ matrix such that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$, then A has a pivot in every column.

(a) True. Proof: Suppose $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) = \vec{0}$, and we need to show that $c_1 = c_2 = 0$. Re-arranging, we have $(c_1 + c_2)\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Since $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent, and we have some linear combination of them equal to $\vec{0}$, all of the coefficients must be 0. So $c_2 = 0$ and $c_1 + c_2 = 0$. Solving gives $c_1 = 0$. This is what we needed to show that $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$ is linearly independent.

(b) False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a pivot in every column, but is not invertible. If A is square, then it is true.

(c) True. $T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0})$. Subtract $T(\vec{0})$ from both sides to get $\vec{0} = T(\vec{0})$.

(d) False. This is equivalent to A being invertible. See solution to #2 on sample midterm.

(e) True. Since $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$, A must have a pivot in every row (theorem 4, p.43). By the Invertible Matrix Theorem, this means that A has a pivot in every column as well (and that A is invertible). Note: this is false if A is not square.

Bonus: **6)** If A is $m \times n$ and B is $n \times r$, then both $\operatorname{Col} B$ and $\operatorname{Nul} A$ are subspaces of $\mathbb{R}^n$. If it happens that $\operatorname{Col} B \subseteq \operatorname{Nul} A$, find the matrix product AB.

Pick any $\vec{x} \in \mathbb{R}^r$. Then $B\vec{x} \in \operatorname{Col} B$ (this is nearly a tautology...can you justify it?). Since $\operatorname{Col} B \subseteq \operatorname{Nul} A$, this means $B\vec{x} \in \operatorname{Nul} A$ as well. Thus $AB\vec{x} = A(B\vec{x}) = \vec{0}$ by the definition of null space. We have seen that for any $\vec{x}$ of the appropriate size, $AB\vec{x} = \vec{0}$. We can compute the j-th column of AB by computing $AB\vec{e}_j$, where $\vec{e}_j$ is the j-th column of the $r \times r$ identity matrix. But $AB\vec{e}_j = \vec{0}$ by the above, so all of the columns of AB are $\vec{0}$. Thus AB is the zero matrix.