

Name: _____

Math 54, Spring 2009, Section 109
Quiz 3 Solutions

[1 - (3 pts)] Let V be the vector space \mathbb{P}_2 of polynomials of degree at most 2, with bases $\mathcal{B} = \{x^2 + 2x - 2, 2x^2 + x, x^2 + 2x - 1\}$ and $\mathcal{C} = \{x^2, x^2 + x, x - 1\}$. Find the change-of-basis matrices ${}_{\mathcal{C} \leftarrow \mathcal{B}} P$ and ${}_{\mathcal{B} \leftarrow \mathcal{C}} P$. (Hint: try to write elements of \mathcal{B} in terms of the elements of \mathcal{C}).

By inspection, $\vec{b}_1 = \vec{c}_1 + 2\vec{c}_3$, $\vec{b}_2 = \vec{c}_1 + \vec{c}_2$, and $\vec{b}_3 = \vec{c}_2 + \vec{c}_3$. Thus ${}_{\mathcal{C} \leftarrow \mathcal{B}} P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$. We can

find ${}_{\mathcal{B} \leftarrow \mathcal{C}} P$ using ${}_{\mathcal{B} \leftarrow \mathcal{C}} P = ({}_{\mathcal{C} \leftarrow \mathcal{B}} P)^{-1}$, so row reduce

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

$$\text{Thus } {}_{\mathcal{B} \leftarrow \mathcal{C}} P = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

[2 - (3 pts)] Suppose you solve a non-homogenous system of 8 linear equations in 10 unknowns, and find that in your solution you have exactly 3 free variables. Is it possible to change the right-hand side of the system of equations (i.e. “change \vec{b} ”) and have the resulting system be inconsistent? (No points for correct answer with incorrect reasoning.)

This situation corresponds to a matrix equation $A\vec{x} = \vec{b}$, where A is 8×10 and has 3 non-pivot columns (i.e. 3 free variables). Thus A only has 7 pivots, and $\text{Rank } A = \dim \text{Col } A = 7$. In particular, $\text{Col } A \neq \mathbb{R}^8$, so there are vectors \vec{c} in \mathbb{R}^8 but not in $\text{Col } A$, for which $A\vec{x} = \vec{c}$ is inconsistent.

[3 - (3 pts)] Let V be a vector space with basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$. Prove **exactly one** of the following two things (if you do both, I'll just grade the first):

(a) If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, then so is $\{[\vec{v}_1]_{\mathcal{B}}, \dots, [\vec{v}_k]_{\mathcal{B}}\}$.

(b) If $\vec{v} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then $[\vec{v}]_{\mathcal{B}} \in \text{Span}\{[\vec{v}_1]_{\mathcal{B}}, \dots, [\vec{v}_k]_{\mathcal{B}}\}$.

(a) Assume that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, and that $x_1[\vec{v}_1]_{\mathcal{B}} + \dots + x_k[\vec{v}_k]_{\mathcal{B}} = \vec{0}$. We need to show that $x_1 = \dots = x_k = 0$. Using the linearity of the coordinate transformation, we have $[x_1\vec{v}_1 + \dots + x_k\vec{v}_k]_{\mathcal{B}} = \vec{0}$. Here we could use the fact that $[\cdot]_{\mathcal{B}}$ is one-to-one, but instead we compute directly from the definition of coordinates that

$$x_1\vec{v}_1 + \dots + x_k\vec{v}_k = 0\vec{b}_1 + \dots + 0\vec{b}_n = \vec{0}.$$

But our original set $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, so we must have $x_1 = \dots = x_k = 0$, which was to be shown.

(b) If $\vec{v} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then there are weights x_1, \dots, x_k such that $\vec{v} = x_1\vec{v}_1 + \dots + x_k\vec{v}_k$. Taking coordinates of both sides, and using the fact that the coordinate transformation is linear, we get

$$\begin{aligned} [\vec{v}]_{\mathcal{B}} &= [x_1\vec{v}_1 + \dots + x_k\vec{v}_k]_{\mathcal{B}} \\ &= x_1[\vec{v}_1]_{\mathcal{B}} + \dots + x_k[\vec{v}_k]_{\mathcal{B}}. \end{aligned}$$

Thus $[\vec{v}]_{\mathcal{B}} \in \text{Span}\{[\vec{v}_1]_{\mathcal{B}}, \dots, [\vec{v}_k]_{\mathcal{B}}\}$.