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Math 54, Spring 2009, Section 109
Quiz 4 Solutions

[**1** - (3 pts)] Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 4 \end{bmatrix}$. (a) Find the eigenvalues of A, and find bases for the corresponding eigenspaces. (b) Is A diagonalizable?

(a) A is upper triangular, so the eigenvalues are 1 and 4 (or 1, 4 and 4, including multiplicity). The corresponding eigenspaces are $\mathrm{Nul}(A-I)$ and $\mathrm{Nul}(A-4I)$. We use row reduction to find bases for these subspaces:

$$[A - I \mid \vec{0}] = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus x_1 is free, and $x_2 = x_3 = 0$. Thus

$$\mathrm{Nul}(A - I) = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is the eigenspace for the eigenvalue 1. Thus $\{(1, 0, 0)\}$ is a basis for this space. Similarly,

$$[[A - 4I \mid \vec{0}] = \begin{bmatrix} -3 & 2 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

Thus we get $x_3 = 0$, x_2 is free, and $x_1 = \frac{2}{3}x_2$. So

$$\mathrm{Nul}(A - 4I) = \left\{ \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \\ 0 \end{bmatrix} : x_2 \in \mathbb{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} \right\},$$

so the eigenspace for the eigenvalue 4 has basis $\{(\frac{2}{3}, 1, 0)\}$. Another choice would be $\{(2, 3, 0)\}$.

(b) A is a 3×3 matrix, with two eigenspaces. Each eigenspace has dimension 1, so the sum of the dimensions of the eigenspaces is less than the size of the matrix. Thus A is not diagonalizable by Theorem 7 on p.324.

[**2** - (3 pts)] Let A be an $n \times n$ matrix, and let $\vec{x} \in \mathbb{R}^n$ be an eigenvector of A with eigenvalue λ. Suppose also that $\vec{x}$ is orthogonal to $A\vec{x}$. What can you say about λ? What does this say about A?

We have $0 = A\vec{x} \cdot x = (\lambda\vec{x}) \cdot \vec{x} = \lambda(\vec{x} \cdot \vec{x})$. Because eigenvectors cannot be $\vec{0}$ by definition, we must have $\vec{x} \cdot \vec{x} \neq 0$. Therefore $\lambda = 0$, which means that A is not invertible.

[**3** - (3 pts)] Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \pi & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$. Find two distinct eigenvectors of A that both have eigenvalue π (the two vectors do *not* need to form a linearly independent set).

We have written $A = PDP^{-1}$, so an eigenvector corresponding to π (the first entry on the diagonal of D) is $P\vec{e}_1$, the first column of P. Why is that? We have $D\vec{e}_1 = \pi\vec{e}_1$, so

$$A(P\vec{e}_1) = PDP^{-1}P\vec{e}_1 = PD\vec{e}_1 = P\pi\vec{e}_1 = \pi P\vec{e}_1.$$

Thus $P\vec{e}_1 = (1,3)$ is an eigenvector of A, with eigenvalue π. Since the eigenvectors of a matrix form a subspace, $2(1,3) = (2,6)$ is another one. Since the eigenvalue π has multiplicity one in this case, the eigenspace corresponding to π is one-dimensional, so the non-zero multiples of $(1,3)$ are the only possible answers.