Name: ______________________________

Math 54, Spring 2009, Section 109
Quiz 5 Solutions

[**1** - (5 pts)] a) Find the general solution of the differential equation $y'' - y' - 2y = t^2 - e^{-t}$.
b) Find the solution of the ODE from part (a) with initial values $y(0) = \frac{1}{4}$ and $y'(0) = -\frac{5}{12}$

(a) The auxiliary equation is $0 = r^2 - r - 2 = (r-2)(r+1)$, so the general solution to the homogenous equation $y'' - y' - 2y = 0$ is $y_h = c_1 e^{2t} + c_2 e^{-t}$. We now try to solve $y'' - y' - 2y = t^2$ by guessing $y_{p_1} = At^2 + Bt + C$ (no extra factor of t, because 0 is not a root of the auxiliary equation). Plugging it in, we get

$$t^2 = y''_{p_1} - y'_{p_1} - 2y_{p_1} = -2A - 2At - B - 2At^2 - 2Bt - 2C = -2At^2 - 2(A+B)t + 2A - B - 2C.$$

Solving gives $A = -\frac{1}{2}$, $B = \frac{1}{2}$, and $C = -\frac{3}{4}$. Next we guess a solution y_{p_2} to $y'' - y' - 2y = -e^{-t}$. Since -1 is a (single) root of the auxiliary equation, we multiply our normal guess by t to get $y_{p_2} = Dte^{-t}$. Plugging it in,

$$-e^{-t} = y''_{p_2} - y'_{p_2} - 2y_{p_2} = D(t-2)e^{-t} + D(t-1)e^{-t} - 2Dte^{-t} = -3De^{-t},$$

so $D = \frac{1}{3}$. Thus the general solution to the ODE is

$$y = y_h + y_{p_1} + y_{p_2} = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{3}{4} + \frac{1}{3}te^{-t}.$$

(b) Plugging in the initial values gives $\frac{1}{4} = y(0) = c_1 + c_2 - \frac{3}{4}$, and $-\frac{5}{12} = y'(0) = 2c_1 - c_2 + \frac{1}{2} + \frac{1}{3}$, which gives the system of equations $c_1 + c_2 = 1$ and $2c_1 - c_2 = -\frac{15}{12}$. Solving gives $c_1 = -\frac{1}{12}$ and $c_2 = \frac{13}{12}$, so the solution to the initial value problem is

$$y(t) = -\frac{1}{12}e^{2t} + \frac{13}{12}e^{-t} - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{3}{4} + \frac{1}{3}te^{-t}.$$

[**2** -(4 pts)] Let $\alpha \in \mathbb{R}$ be a constant, and consider the following initial value problem

$$(x+1)y'' - (x^2-1)y' = (\alpha - x)\sqrt{-x},$$
$$y(-\tfrac{1}{3}) = 23, \qquad y'(-\tfrac{1}{3}) = 47.$$

What is the largest interval for which Theorem 1 in section 6.1 guarantees a solution? (Hint: it will depend on the value of the parameter α).

Putting the equation in standard form, we get

$$y'' - (x-1)y' = \frac{\alpha - x}{x+1}\sqrt{-x}.$$

Because of the square root, we must have $x \leq 0$. If $\alpha = -1$, then $\frac{\alpha - x}{x+1} = -1$, so there are no other restrictions and the largest open interval containing $-\frac{1}{3}$ on which all of the coefficients and the right hand side are continuous is $(-\infty, 0)$. If $\alpha \neq -1$, then $\frac{\alpha - x}{x+1}$ blows up at $x = -1$, so the largest such interval is $(-1, 0)$.