Name: ______________________________

Math 54, Spring 2009, Section 109
Quiz 7 Solutions

[**1** - (4 pts)] Consider the PDE

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

If $u(x,t) = R(r)T(\theta)$ were to satisfy this PDE, what ODEs would R and T have to satisfy? (Hint: this is just separation of variables).

Plugging u into the PDE yields

$$TR'' + \frac{1}{r}TR' + \frac{1}{r^2}T''R = 0.$$

Rearranging gives $T(r^2R'' + rR') = -T''R$, or

$$-\frac{T''}{T} = \frac{r^2R'' + rR'}{R}.$$

Since the left side of this equality depends only on θ, and the right side only on r, they must both be identically equal to some constant K. We then have

$$\begin{cases} T'' + KT = 0, \\ r^2R'' + rR' - KR = 0. \end{cases}$$

[**2** -(5 pts)] (a) Compute the Fourier series of $f(x) = |x|$ on the interval $[-\pi, \pi]$. Sketch a graph of the function the Fourier series converges to. (This function should be defined for all $x \in \mathbb{R}$.)
(b) What is the Fourier cosine series of $g(x) = x$ on the interval $[0, \pi]$?

Since f is an even function, the coefficient b_n of $\sin(nx)$ will be 0 for all n. Using the fact that $|x|\cos(nx)$ is an even function, we can compute the coefficients a_n:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x dx = \pi,$$

and for $n \neq 0$:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx dx \\
&= \frac{2}{\pi}\int_0^{\pi} |x| \cos nx dx \\
&= \frac{2}{\pi}\int_0^{\pi} x \cos nx dx \\
&= \frac{2}{\pi}\left[\frac{1}{n} x \sin nx\right]_0^{\pi} - \frac{2}{\pi}\left(\frac{1}{n}\int_0^{\pi} \sin nx dx\right) \\
&= \frac{2}{\pi n^2}((-1)^n - 1).
\end{aligned}$$

We then have that the Fourier series of f is

$$\begin{aligned}
f(x) &\sim \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}((-1)^n - 1)\right)\cos(nx) \\
&= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2}\cos{(2k+1)x}.
\end{aligned}$$

Since $|x|$ is continuous on $(-\pi, \pi)$, the Fourier series to $|x|$ on this interval. We also have $|-\pi| = |\pi| = \pi$, so the Fourier series converges to π at both endpoints. So the Fourier series converges to the 2π-periodic extension of $f(x) = |x|$ on $[-\pi, \pi]$. This is a "triangular wave."

(b) The even extension of $g(x) = x$ on $[0, \pi]$ is $f(x) = |x|$ on $[-\pi, \pi]$. Thus the Fourier cosine series of g is the same as the Fourier series of f computed in part (a).