**Math 54, Spring 2009, Sections 109 and 112**
**Worksheet 2 (Lay 1.7-1.8) Solutions**

**(1)** Classify the following sets as linearly independent or linearly dependent. (Hint: no calculations needed).

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ - *Linearly dependent, because it contains* $\vec{0}$ *(Theorem 9, p.69).*

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$ - *Linearly dependent, because you have more vectors than entries in each vector (Theorem 8, p.69).*

(c) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 9 \\ -12 \end{bmatrix} \right\}$ - *Linearly independent, because we have a two element set with neither vector a multiple of the other (bottom p.67)*

**(2)** True/False: If it's true, give a justification. If it's false, give a counterexample.

(a) If $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a linearly independent set of vectors, and $A$ is a matrix, then $\{A\vec{v}_1, \ldots, A\vec{v}_p\}$ is also linearly independent.

*False.*

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

*Note: there are certainly smaller counterexamples.*

(b) If $\vec{b}$ is in the span of the columns of $A$, then $A\vec{x} = \vec{b}$ is consistent.
*True. See pages 42-43.*

**(3)** (#39 from p.72) Suppose $A$ is a $m \times n$ matrix with the property that for all $\vec{b}$ in $\mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ has at most one solution. Explain why the columns of $A$ are linearly independent.

*In particular, we can consider when* $\vec{b} = \vec{0}$*. In this case, by our assumption we know that* $A\vec{x} = \vec{0}$ *has at most one solution. But we already know that there is at least one solution to* $A\vec{x} = \vec{0}$*, namely the trivial solution. So our assumption means that* $A\vec{x} = \vec{0}$ *must have* `only` *the trivial solution. Thus the columns of* $A$ *are linearly independent (see bottom p. 66).*

**(4)** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v}_1, \ldots \vec{v}_p\}$ be a linearly dependent set in $\mathbb{R}^n$. Assume that $T(\vec{v}_i) \neq T(\vec{v}_j)$ when $i \neq j$. Show that $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is linearly dependent in $\mathbb{R}^m$. (Compare to 2a).

*Since* $\{\vec{v}_1, \ldots, \vec{v}_p\}$ *is linearly dependent, there exist scalars* $x_1, \ldots, x_p$ *such that*

$$x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0},$$

*where not all of the* $x_j$ *are 0. Then, by the linearity of* $T$*, we have*

$$\begin{aligned} \vec{0} &= T(\vec{0}) \quad \textit{(See (3) on p.77)} \\ &= T(x_1\vec{v}_1 + \cdots x_p\vec{v}_p) \\ &= x_1T(\vec{v}_1) + \cdots x_pT(\vec{v}_p) \quad \textit{(By linearity of } T\textit{.)} \end{aligned}$$

*By the definition of linear dependence, this means that* $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ *is linearly dependent, which is what we were trying to show.*