Math 54, Spring 2009, Sections 109 and 112
(Mini) Worksheet 6 Solutions (Lay 6.5)

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

(i) Find $\text{Proj}_W \vec{b}$.

(ii) Why is $A\vec{x} = \text{Proj}_W \vec{b}$ consistent?

(iii) Solve $A\vec{x} = \text{Proj}_W \vec{b}$.

(iv) If x_0 is a solution from (iii), why is $\left\| A\vec{x}_0 - \vec{b} \right\| \leq \left\| A\vec{x} - \vec{b} \right\|$ for any $x \in \mathbb{R}^2$?

(i) If $\vec{u} = (1, 2)$, then $\{\vec{u}\}$ is an orthoogonal basis for W. So the formula for $\text{Proj}_W \vec{b}$ is

$$\text{Proj}_W \vec{b} = \frac{\vec{u} \cdot \vec{b}}{\vec{u} \cdot \vec{u}} \vec{u} = (\frac{3}{5}, \frac{6}{5}).$$

(ii) In this case, $W = \text{Col}\, A$, so $\text{Proj}_W \vec{b} \in \text{Col}\, A$. An equation $A\vec{x} = \vec{c}$ is consistent if and only if $\vec{c} \in \text{Col}\, A$, so $A\vec{x} = \text{Proj}_W \vec{b}$ is consistent.

(iii) Forming the augmented matrix and row-reducing we get

$$\begin{bmatrix} 1 & 2 & 3/5 \\ 2 & 4 & 6/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

so in general we have x_2 free and $x_1 = \frac{3}{5} - 2x_2$. Alternatively, this could be written in parametric vector form $(3/5, 0) + x_2(-2, 1)$ (with x_2 free again).

(iv) The Best Approximation Theorem says that $\left\| \text{Proj}_W \vec{b} - \vec{b} \right\| < \left\| v - \vec{b} \right\|$ for any $v \in W$ (that is, any $v \in \text{Col}\, A$) as long as $v \neq \text{Proj}_W \vec{b}$. The elements of $\text{Col}\, A$ are precisely those of the form $A\vec{x}$ for any $x \in \mathbb{R}^2$, and we chose x_0 so that $Ax_0 = \operatorname{Proj}_W \vec{b}$. Substituting these into the above we have $\left\|\operatorname{Proj}_W \vec{b} - \vec{b}\right\| \leq \left\|A\vec{x} - \vec{b}\right\|$ for any $\vec{x} \in \mathbb{R}^2$. Why "$\leq$" as opposed to "$<$"? The system $A\vec{x} = \vec{b}$ has a free variable, so if $x_1 \neq x_0$ is another solution to $A\vec{x} = \operatorname{Proj}_W \vec{b}$, then $\left\|\operatorname{Proj}_W \vec{b} - \vec{b}\right\| = \left\|Ax_1 - \vec{b}\right\|$.