Name:____

Math 54 Final Exam Solutions

August 14, 2009

Instructor: James Tener

Problem 1: ______ / 20 points

Problem 2: ______ / 10 points

Problem 3: ______ / 15 points

Problem 4: ______ / 10 points

Problem 5: ______ / 10 points

Problem 6: ______ / 15 points

Problem 7: ______ / 5 points

Total: ______ / 85 points

Instructions:

- Answer in the space provided. If you run out of space, I can give you more paper.
- Show all of your work. When justifying answers, express yourself clearly and in an organized fashion. You are graded on what you write down, not what you mean to say.
- You may cite theorems from class/the book by (correctly) stating what it says.
- Cross out any work you do not want graded.
- No calculators are allowed.

Problem 1. You do not need to justify your answers on this problem. (5 points each)

(a) Give an example of a matrix A and a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a non-unique solution.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(b) Define what is meant by a <u>fundamental matrix</u> for the system of ODEs $\vec{x}\,' = A\vec{x}$. (Be sure to define it, not explain how to find it)

A fundamental matrix is a matrix whose columns form a basis for the solution space of the system.

(c) Give an example of an infinite dimensional vector space.

Possible examples: the vector space of all polynomials, all continuous functions on an interval, or all functions from $\mathbb{R}$ to $\mathbb{R}$.

(d) Let A be an $m \times n$ matrix, and let $\cdot$ be the dot product. Suppose $x = (Ay) \cdot z$ (note: vector arrows have been omitted). For each of x, y and z, say whether it is a vector or a scalar, and say how many entries each vector has.

For Ay to be defined, we need $\vec{y} \in \mathbb{R}^n$. Then $A\vec{y} \in \mathbb{R}^m$, so $\vec{z} \in \mathbb{R}^m$. The dot product outputs a scalar, so $x \in \mathbb{R}$.

Problem 2. Find all solutions to the ODE $y'' - 4y' + 5y = te^{2t}$ such that $y(0) = y'(0)$. (10 points)

The auxiliary equation is $r^2 - 4r + 5 = 0$, which has roots $2 \pm i$. Thus the general solution to the homogenous equation is $c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$. We now guess $y_p = (At + B)e^{2t}$. Plugging this into the ODE yields $A = 1$ and $B = 0$, so the general solution to the (nonhomogenous) ODE is $y(t) = te^{2t} + c_1 e^{2t} \cos t + c_2 e^{2t} \sin t$. Setting $y(0) = y'(0)$ we get $c_1 = 1 + 2c_1 + c_2$, or $c_2 = -(1 + c_1)$. So the general solution to our problem is $e^{2t} + c_1 e^{2t} \cos t - (1 + c_1)e^{2t} \sin t$.

Problem 3. Say whether the given statement is true or false. If it is true, explain why. If it is false, provide a counterexample showing that it is false. No points are given for true/false without correct justification (i.e. **no points for "false" without a concrete counterexample!**). (3 points each)

(a) The initial value problem $ay'' + by' + cy = 0$, $y(0) = 0$ has a unique solution for every $a, b, c \in \mathbb{R}$.

False. Setting $a = 1$ and $b = c = 0$, we get a whole family of solutions $y(t) = \alpha t$ for any $\alpha \in \mathbb{R}$.

(b) If A is $n \times n$ and $A^2 = 0$ (zero matrix), then the characteristic polynomial of A is λ^n.

False. E.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has characteristic polynomial $-\lambda^3$. However, if $A^2 = 0$, then all of the eigenvalues of A are 0, so the characteristic polynomial is $(-1)^n \lambda^n$.

(c) If A can be row reduced to B, then A and B have the same determinant.

False. E.g. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) If A is 5×6 and $\dim \operatorname{Nul} A = 1$, then $T(\vec{x}) = A\vec{x}$ is onto.

True. The rank-nullity theorem says that $\operatorname{Rank} A = 5$. Since $\operatorname{Ran} T \subseteq \mathbb{R}^5$ and $\dim \operatorname{Ran} T = 5$, we have $\operatorname{Ran} T = \mathbb{R}^5$ and T is onto.

(e) Every linearly dependent subset of a finite dimensional vector space V has a subset that spans V.

False. $\left\{\vec{0}\right\}$ is a linearly dependent subset of $\mathbb{R}^2$ (say), but certainly no subset spans.

Problem 4. (a) Let $f(x) = x^2 + 1$, defined on the interval $[0, \pi]$. Find the Fourier sine series of f, and graph the function it converges to on the interval $[-\pi, \pi]$. Be sure to indicate on the graph the value of the sine series at all points in $[-\pi, \pi]$. You do not have to evaluate any integrals. Your answer may have coefficients c_n in it, along with formula(s) $c_n = \cdots \int_a^b \cdots \, dx$. (6 points)

On $(-\pi, 0) \cup (0, \pi)$, the Fourier sine series converges to the *odd* extension of f. At $\pm\pi$ and 0 it converges to 0. (Graph omitted). The Fourier sine series is $\sum_{n=1}^{\infty} b_n \sin nx$ where

$$b_n = \frac{2}{\pi} \int_0^{\pi} (x^2 + 1) \sin nx.$$

(b) Find a formal solution to the heat problem given below. Again, you do not need to evaluate any integrals, but you should provide the formula for any fundamental solutions u_n that you use. (4 points)

$$\begin{cases} u_t = 3u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi, t) = 0 & t > 0, \\ u(x,0) = x^2 + 1 & 0 < x < \pi. \end{cases}$$

A formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2 t} \sin nx$$

where b_n is as above.

Problem 5. If u is a function of the variables x and t, consider the PDE $u_{xx} + u_t + u_{tt} = 0$.

(a) If $u(x,t) = X(x)T(t)$ solves the PDE, derive ODEs (sharing a common constant) that X and T would have to satisfy. (4 points)

If u were to satisfy this PDE, we would have $X''T + XT' + XT'' = 0$. That is,

$$-\frac{X''}{X} = \frac{T'' + T'}{T}.$$

Since the left side depends on x and the right side depends on t, they must be equal to a common constant, K. This yields

$$\begin{cases} X'' + KX = 0, \\ T'' + T' - KT = 0. \end{cases}$$

(b) Use (a) to give an example of a non-trivial solution to the PDE. (6 points)

We may pick *any* value of K, so long as the two ODEs have non-trivial solutions. Just for fun, pick $K = 47$. Then $\cos\sqrt{47}x$ solves the first ODE, and e^{rt} solves the second ODE, where

$$r = \frac{-1 + \sqrt{1 + 4 * 47}}{2}.$$

One can now check that

$$u(x,t) = \cos(\sqrt{47}x)e^{rt}$$

solves the PDE.

Problem 6. Let V be the vector space of real 2×2 matrices, and define the function $\mathrm{tr} : V \to \mathbb{R}$ by

$$\mathrm{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

That is, the tr function outputs the sum of the entries on the diagonal. Define an inner product on V by $\langle A, B\rangle = \mathrm{tr}(AB^t)$. You may assume this has all of the properties of an inner product except the ones you are asked to prove in part (a).

(a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Show that $\langle A, B\rangle = \langle B, A\rangle$. Also show that if $\langle A, A\rangle = 0$, then A is the zero matrix. (4 points)

Observe that $\mathrm{tr}(C) = \mathrm{tr}(C^t)$ for any matrix C, so that $\langle A, B\rangle = \mathrm{tr}(AB^t) = \mathrm{tr}((AB^t)^t) = \mathrm{tr}(BA^t) = \langle B, A\rangle$. Or, one can verify directly that $\langle A, B\rangle = ax + by + cz + dw = \langle B, A\rangle$. Thus $\langle A, A\rangle = a^2 + b^2 + c^2 + d^2$, so $\langle A, A\rangle = 0$ if and only if $a = b = c = d = 0$.

(b) Prove that $|ax + by + cz + dw| \leq \sqrt{(a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + w^2)}$ for any real numbers x, y, z, w, a, b, c, d. (5 points)

The Cauchy-Schwarz inequality says that $|\langle A, B\rangle| \leq \|A\|\|B\|$. Plugging in from (a), we get the desired inequality.

Problem 6 continued.

(c) If $W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\}$ is the subspce of V consisting of symmetric matrices, find a basis for $W^\perp$ (with respect to the given inner product). (6 points)

We are looking for matrices $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that

$$\langle \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \begin{bmatrix} x & y \\ z & w \end{bmatrix} \rangle = 0$$

for *every* $a, b, c \in \mathbb{R}$ (so that B is orthogonal to every element of W). Plugging in, we need $ax + by + bz + cw = 0$ for all $a, b, c \in \mathbb{R}$. That is $ax + b(y + z) + cw = 0$. Since we can choose a, b, c to be anything, we need $x = w = 0$, and $y = -z$. Thus

$$W^\perp = \left\{ \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} : y \in \mathbb{R} \right\}.$$

A basis for this subspace is $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$.

Problem 7. You are a retired secret agent who likes to work on math puzzles. Let

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

where x_{ij} is an **integer** for every i, j. Assume that all of the diagonal entries in A are odd, and that all of the non-diagonal entries are even. That is, x_{ij} is odd if and only if $i = j$. Is A always invertible, sometimes invertible, or never invertible? (5 points)

Expanding $\det A$ across the first row, we get

$$\begin{aligned} \det A &= x_{11} \det \begin{bmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{bmatrix} - x_{12} \det \begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix} + x_{13} \det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \\ &= x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12} \det \begin{bmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{bmatrix} + x_{13} \det \begin{bmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}. \end{aligned}$$

Since the product of two numbers is odd if and only if both of the numbers are odd, we see that the above expression is an odd number plus a bunch of even numbers. Thus the determinant of A is odd, and in particular isn't 0. So A is always invertible.