Math 54, Summer 2009, Lecture 4
Midterm 1 Review

Note: this sheet is not exhaustive. It's just a collection of some things that seemed important to me as I was writing it. It is not a substitute for studying.

Important terms

The following definitions are important to know. It's not enough to be able to recite definitions, you have to understand how they're used, and how they fit together (i.e. a matrix cannot be linearly independent, a matrix doesn't have a solution, $\mathbb{R}^2$ is not contained in $\mathbb{R}^3$, etc.) and to be comfortable using them.

- Basics: echelon form, reduced echelon form, pivots, augmented matrix, free variable, basic variable (see sections 1.1 and 1.2)
- A *square* matrix A is called invertible if there is another matrix B of the same size such that $AB = BA = I_n$.
- An elementary matrix is a matrix that can be obtained from the identity matrix with a single elementary row operation.
- A collection of vectors $\vec{v}_1, \dots, \vec{v}_p$ in $\mathbb{R}^n$ is called linearly independent if you cannot have $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$ unless $c_1 = c_2 = \cdots = c_p = 0$. This is equivalent to $A\vec{x} = \vec{0}$ having only the trivial solution $\vec{x} = 0$, where $A = [\vec{v}_1 | \cdots | \vec{v}_p]$. (Why are these equivalent?)
- Alternatively, $\vec{v}_1, \dots, \vec{v}_p$ is linearly dependent if there are some $c_1, \dots, c_p$, not all 0, such that $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$. This is equivalent to $A\vec{x} = \vec{0}$ having a non-trivial solution, where A is as above.
- Given vectors $\vec{v}_1, \dots, \vec{v}_p$ in $\mathbb{R}^n$, the span of these vectors is the set $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$, the set of all linear combinations of the given vectors.
- If $H = \text{Span}\, \vec{v}_1, \dots \vec{v}_p$, then we say that $\{\vec{v}_1, \dots, \vec{v}_p\}$ spans H, and that $\{\vec{v}_1, \dots, \vec{v}_p\}$ generate H.
- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it respects addition and scalar multiplication. That is, for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and every $c \in \mathbb{R}$ we have $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(c\vec{x}) = cT(\vec{x})$.

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called <u>one-to-one</u> if for every $y \in \mathbb{R}^m$, there is at most one $x \in \mathbb{R}^n$ satisfying $T(\vec{x}) = \vec{y}$. That is, no two vectors in $\mathbb{R}^n$ get sent to the same place.

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called <u>onto</u> if for every $y \in \mathbb{R}^m$, there is at least one $x \in \mathbb{R}^n$ satisfying $T(\vec{x}) = \vec{y}$. That is, everything in $\mathbb{R}^m$ gets hit by something.

- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the <u>domain</u> of T is $\mathbb{R}^n$, and the <u>codomain</u> of T is $\mathbb{R}^m$.

- If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$, then $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is the <u>composition</u> of S and T, given by $(S \circ T)(\vec{x}) = S(T(\vec{x}))$.

- The <u>standard matrix</u> of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the unique $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for every $x \in \mathbb{R}^n$. The columns of the standard matrix are $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, where e_j is the j-th column of I_n.

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called <u>invertible</u> if there is some $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(S(\vec{x})) = S(T(\vec{x})) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^n$. Note how this parallels the definition of invertible matrix.

An inexhaustive list of things you should know how to do: solve systems of linear equations and express the solution as free variables and basic variables or in parametric vector form, row reduce a matrix, multiply two matrices (and know if the multiplication is defined), determine if a matrix is invertible and find it's inverse (if applicable), find the standard matrix of a linear transformation, understand the relationship between matrix equations and vector equations and translate between the two, determine if a set is linearly independent and if its not find linear dependences, find determinants, understand the "onto" and "one-to-one" theorems for $m \times n$ matrices, and how they relate to the invertible matrix theorem for $n \times n$ matries.