Math 54, Summer 2009, Lecture 4
Midterm 1 Review Exercises Solutions

These exercises don't cover some of the **very important** computational-type problems, including many of the things listed under the "be able to" section of the review sheet. You can find examples of those types of problems on the sample exam and in the sections of the book (including the supplemental exercises at the end of each chapter). These are a little more theoretical, and are aimed at making sure you have a good grasp of the ideas underlying the algorithms.

1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and A be its standard matrix. Complete the following table so that statements in the same row are equivalent:

Property of T	Columns of A	Pivots of A	$A\vec{x} = \vec{b}$?
One-to-one	Linearly independent	Every column	≤ 1 solution for every $\vec{b}$
Onto	Span $\mathbb{R}^m$	Every row	≥ 1 solution for every $\vec{b}$

2) Let A be a 17×17 matrix such that $A^{12} = I_{17}$. Explain why $A\vec{x} = \vec{0}$ only when $\vec{x} = \vec{0}$.

Note that $I_{17} = A^{12} = A^{11}A = AA^{11}$. Thus A is invertible (and $A^{-1} = A^{11}$). By the invertible matrix theorem, $A\vec{x} = \vec{0}$ has only the trivial solution.

3) (#10, p.184) Suppose A is an invertible square matrix. Explain why A^TA is also invertible, and then show that $A^{-1} = (A^TA)^{-1}A^T$.

Recall that $\det A = \det A^T$, so $\det A^TA = \det A^T \det A = (\det A)^2$. Since A is invertible, $\det A \neq 0$, and thus $(\det A)^2 \neq 0$. Hence A^TA is invertible by a theorem from class.

We can now check that

$$\left((A^TA)^{-1}A^T\right)A = (A^TA)^{-1}(A^TA) = I,$$

so by the invertible matrix theorem $A^{-1} = (A^TA)^{-1}A^T$ (we used the invertible matrix theorem to not have to check that $A(A^TA)^{-1}A^T = I$ as well).

4) Suppose you have a square matrix such that $A^3 = 0$ (the zero matrix). Use matrix algebra to compute $(I - A)(I + A + A^2)$. Generalize to show that if $A^k = 0$ for some $k \geq 1$, then $(I - A)$ is invertible.

We compute

$$(I - A)(I + A + A^2) = I + A + A^2 - A - A^2 - A^3 = I - A^3 = I.$$

Similarly, if $A^k = 0$, then $(I - A)(I + A + A^2 + \cdots + A^{k-1}) = I - A^k = I$, so by the invertible matrix theorem $I - A$ is invertible.

5) True or False? If true, justify. If false, provide a counterexample. (Some of these are from p.102.)

(a) If $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$, so is $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$.

(b) If an $m \times n$ matrix A has a pivot in every column or has a pivot in every row, then it is invertible.

(c) If T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

(d) If A is a square matrix, then it can be written as a product of elementary matrices.

(e) If A is an $n \times n$ matrix such that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$, then A has a pivot in every column.

(a) True. Suppose $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) = \vec{0}$. Then $(c_1 + c_2)\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. But $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent, so $c_1 + c_2 = 0$ and $c_2 = 0$. It follows that $c_1 = c_2 = 0$. Thus $\vec{v}_1$ and $\vec{v}_2$ have no (non-trivial) linear dependences, which means that the set is linearly independent by definition. Just for fun: can you think of a way to solve this problem using elementary matrices and invertibility?

(b) False. This is only true if A has a pivot in every column *and* a pivot in every row. Counterexample: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has a pivot in every row, but is not square (so not invertible).

(c) True. If T is linear, $T(\vec{0}) = T(\vec{0}+\vec{0}) = T(\vec{0}) + T(\vec{0})$. Subtracting $T(\vec{0})$ from both sides yields $T(\vec{0}) = \vec{0}$.

(d) False. Elementary matrices are invertible, so products of elementary matrices are as well. So $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which is not invertible because it does not have a pivot in every row, cannot be written as a product of elementary matrices.

(e) True, by the invertible matrix theorem. If $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$, then A has a pivot in every row. A is square, so this implies there is a pivot in every column.

Bonus: **(6)** Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. (a) Factor A^{-1} as a product of elementary matrices. That is, find elementary matrices $E_1, \ldots, E_m$ such that $A^{-1} = E_1 \cdots E_m$. (b) Use this to factor A as a product of elementary matrices.

Let's expand A^{-1} using row reduction:

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is the same as

$$A \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \to \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \to \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \to$$

$$\to \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = I_2.$$

Thus $A^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) Using the rule for the inverse of a product of matrices, we get

$$A = (A^{-1})^{-1} = \left(\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} =$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$