Name: **Solutions**

Math 54 Midterm 1

July 10, 2009

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Problem 1: ______ / 10 points

Problem 2: ______ / 10 points

Problem 3: ______ / 10 points

Problem 4: ______ / 10 points

Problem 5: ______ / 10 points

Problem 6: ______ / 10 points

Instructions:

- Answer in the space provided. If you run out of space, I can give you more paper.
- Show all of your work. When justifying answers, express yourself clearly and in an organized fashion. You are graded on what you write down, not what you mean to say.
- You may cite theorems from class/the book by (correctly) stating what it says.
- Cross out any work you do not want graded.
- No calculators are allowed.

Problem 1. Let A be an $m \times n$ matrix, and let $T(\vec{x}) = A\vec{x}$ be the associated linear transformation. No justifications are necessary for this problem, but be sure to use all technical terms appropriately.

(a) What is the domain of T? What is the codomain of T? (2 points)

Domain is $\mathbb{R}^n$, codomain is $\mathbb{R}^m$.

(b) State three conditions (on A or T) that are equivalent to T being one-to-one. (3 points)

- $A\vec{x} = \vec{0}$ has only the trivial solution
- The columns of A are linearly independent
- A has a pivot in every column

(c) State three conditions (on A or T) that are equivalent to T being onto. (3 points)

- The columns of A span $\mathbb{R}^m$
- $A\vec{x} = \vec{b}$ is consistent for every $b \in \mathbb{R}^n$
- A has a pivot in every row

(d) State the formula used to find A, in terms of T. (2 points)

$$A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}.$$

Problem 2. Determine whether the following set is linearly dependent or linearly independent (10 points)

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ 8 \end{bmatrix} \right\}$$

The columns of a matrix are linearly independent if and only if there is a pivot in every column, so we row reduce (steps omitted)

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & -2 & 4 \\ 4 & 1 & -1 & 8 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see that the matrix does not have a pivot in the last column, so the columns are linearly dependent.

Problem 3. Say whether the given statement is true or false. **If it is true, explain why. If it is false, provide a counterexample showing that it is false or correct the statement.** No points are given for true/false without correct justification. (2 points each)

(a) If A and B are $m \times n$, then both AB^T and $A^T B$ are defined.

True. A^T is $n \times m$ and B is $m \times n$, so the product is defined. Similarly for A and B^T.

(b) If A is an invertible matrix such that $A = A^{-1}$, then $\det A = 1$.

False. If $A = A^{-1}$, then $A^2 = I$. Thus $1 = \det(A^2) \det(A)^2$. So the correct statement is that if A is an invertible matrix such that $A = A^{-1}$, then $\det A = \pm 1$. Or a counter example would be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(c) If $n \times n$ matrices A and B have the property that $AB = I_n$, then it must be the case that $AB = BA$.

True. The invertible matrix theorem says that if A and B are square and $AB = I_n$, then $BA = I_n$ (and $B = A^{-1}$).

(d) If $A\vec{x} = \vec{0}$ has the trivial solution, then the columns of A are linearly independent.

False. If $A\vec{x} = \vec{0}$ has **only** the trivial solution, then the columns of A are linearly independent.

(e) If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

False. The elementary row operations that take $AtoI_n$ take I_n to A^{-1}, not the other way around. Or, if you'd like a counter example:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{but on the inverse} \quad \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 4. Let $A = \begin{bmatrix} 0 & k-1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & k+1 \end{bmatrix}$.

(a) Use $\det A$ to determine all values of k for which A is invertible. (4 points).

A standard calculation gives $\det A = -k(k-1)$. Since A is invertible if and only if $\det A \neq 0$, this means that A is invertible precisely when $k \neq 0, 1$.

(b) Consider the linear system

$$A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where A is as above. For every value of k, say how many solutions this matrix equation has. Hint: part (a) gives the answer for most values of k, leaving you with just a few to check. (6 points)

When A is invertible, $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$. Thus for $k \neq 0, 1$ there is exactly one solution.

When $k = 0$, we can check that the augmented matrix of the system $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there is no equation like $0 = b$ for $b \neq 0$, and the coefficient matrix has a non-pivot column, the system is consistent and has a free variable. Thus when $k = 0$, there are infinitely many solutions.

For $k = 1$, we have

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

which has an equation like $0 = -1$, and is therefore inconsistent. Thus there is one solution when $k \neq 0, 1$, infinitely many solutions when $k = 0$, and no solutions when $k = 1$.

Problem 5. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be the function given by

$$T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \det \begin{bmatrix} x_1 & 2 & 1 \\ x_2 & 1 & -1 \\ x_3 & 0 & 1 \end{bmatrix}.$$

That is, T takes a vector in $\mathbb{R}^3$ as input, puts it in as the first column of a matrix, and returns the determinant of that matrix.

(a) Show that T is a linear transformation. Hint: use cofactor expansion on the first column, and don't work too hard. (6 points)

Option 1: We can show directly that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^3$:

$$\begin{aligned} T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) &= \det \begin{bmatrix} x_1 + y_1 & 2 & 1 \\ x_2 + y_2 & 1 & -1 \\ x_3 + y_3 & 0 & 1 \end{bmatrix} \\ &= (x_1 + y_1) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (x_2 + y_2) \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + (x_3 + y_3) \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ &= (x_1 + y_1) - 2(x_2 + y_2) - 3(x_3 + y_3) \\ &= (x_1 - 2x_2 - 3x_3) + (y_1 - 2y_2 - 3y_3) \\ &= T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) + (\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}) \end{aligned}$$

and

$$\begin{aligned} T(c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) &= \det \begin{bmatrix} cx_1 & 2 & 1 \\ cx_2 & 1 & -1 \\ cx_3 & 0 & 1 \end{bmatrix} \\ &= cx_1 - 2cx_2 - 3cx_3 \\ &= cT(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}). \end{aligned}$$

Thus we have shown that T is a linear transformation.

Option 2: We can calculate

$$\begin{aligned}
T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) &= \det \begin{bmatrix} x_1 & 2 & 1 \\ x_2 & 1 & -1 \\ x_3 & 0 & 1 \end{bmatrix} \\
&= x_1 - 2x_2 - 3x_3 \\
&= \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.
\end{aligned}$$

This means that T is a matrix transformation, and therefore T is linear.

(b) What is the standard matrix of T? (4 points)

Following option 1, we can calculate $T(\vec{e}_1) = 1$, $T(\vec{e}_2) = -2$ and $T(\vec{e}_3) = -3$, so the standard matrix of T is $\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$. Alternatively, following option 2 we can see (from the definition) that this is the standard matrix of T.

Problem 6. You are a secret agent. To accomplish your mission, it is imperative that you find all solutions to the matrix equation $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Unfortunately, you were detected while you were trying to find out what the entries of A were, and the only information you recovered was that

$$A^{-1} = \begin{bmatrix} \pi & \sqrt{2} & 47 & * & \pi \\ 47 & * & -22 & \Gamma & -22 \\ -1 & -2 & -3 & * & -4 \\ 23 & * & 13 & 47 & -11 \\ 0 & \Omega & 66 & 24 & 19 \end{bmatrix},$$

where the $*$'s represent numbers that were unreadable due to a coffee stain. Report as much as you can about the existence and/or values of solution(s) to $A\vec{x} = \vec{b}$. (10 points, and the fate of the world)

Since A is invertible, $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$, and that solution is

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 47 \\ 47 \\ 0 \\ 47 \\ 47 \end{bmatrix}.$$