Math 54, Summer 2009, Lecture 4
Midterm 2 Review Exercises

These exercises don't cover some of the **very important** computational-type problems, including many of the things listed under the "be able to" section of the review sheet. You can find examples of those types of problems on the sample exam and in the sections of the book (including the supplemental exercises at the end of each chapter). These problem are a little more theoretical, and are aimed at making sure you have a good grasp of the ideas underlying the algorithms.

1) (a) Assume that A is an $m \times n$ matrix and that B is an $n \times p$ matrix. Show that $AB = 0$ (the zero matrix) if and only if $\operatorname{Col} B \subseteq \operatorname{Nul} A$.

(b) What can you say about $\operatorname{Rank} A$ and $\operatorname{Rank} B$?

2) [p.371, #3] Suppose $\vec{x}$ is an eigenvector of A corresponding to an eigenvalue λ. Show that $\vec{x}$ is an eigenvector of $5I - 3A + A^2$. What is its eigenvalue?

3) Find a 2×2 matrix A such that $A^2 + 6I = 5A$. What if we require that A not be diagonal?

4) [p.371, #1] True or false? If true, explain why, and if false provide a counterexample. Assume all matrices are square.

(a) If A contains a row of zeros, then 0 is an eigenvalue of A.

(b) Every eigenvector of A is also an eigenvector of A^2.

(c) If A is diagonalizable, then the columns of A are linearly independent.

(d) If A and B are invertible $n \times n$ matrices, then AB is similar to BA.

(e) If A is an $n \times n$ diagonalizable matrix, then every vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A.

5) (a) Suppose that A is an $n \times m$ matrix. Show that $A^T A\vec{x} \cdot \vec{x} \geq 0$ for every $\vec{x} \in \mathbb{R}^m$.

(b) Show that if $\|\vec{x}\| \leq 1$, then $\|A\vec{x}\|^2 \leq \left\|A^T A\vec{x}\right\|$.

6) Suppose that $\vec{y} \in \mathbb{R}^n$, and that W is a subspace of $\mathbb{R}^n$. Show that $\vec{y} = \operatorname{Proj}_W \vec{y} + \operatorname{Proj}_{W^\perp} \vec{y}$ and that $\|\vec{y}\|^2 = \|\operatorname{Proj}_W \vec{y}\|^2 + \|\operatorname{Proj}_{W^\perp} \vec{y}\|^2$.

(7) Let V be the inner product space $C[0,1]$ of all continuous functions defined on the interval $[0,1]$, with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Find a function that is orthogonal to both $f_1(x) = 1$ and $f_2(x) = x^3$.

(8) Let $\mathcal{B}$ be a basis for a vector space V. Use the fact that $[\cdot]_B$ is an isomorphism to prove that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ spans V if and only if $\{[\vec{v}_1]_B, \ldots, [\vec{v}_n]_B\}$ spans $\mathbb{R}^m$.