Name: **Solutions**

Math 54 Midterm 2

July 31, 2009

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Problem 1: ______ / 15 points

Problem 2: ______ / 10 points

Problem 3: ______ / 15 points

Problem 4: ______ / 10 points

Problem 5: ______ / 10 points

Problem 6: ______ / 10 points

Total: ______ / 70 points

Instructions:

- Answer in the space provided. If you run out of space, I can give you more paper.
- Show all of your work. When justifying answers, express yourself clearly and in an organized fashion. You are graded on what you write down, not what you mean to say.
- You may cite theorems from class/the book by (correctly) stating what it says.
- Cross out any work you do not want graded.
- No calculators are allowed.

Problem 1.

(a) Let V be a vector space, and let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a subset of V. Define what it means for S to be linearly independent, and what it means for S to span V. (4 points)

S is linearly independent means that whenever $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$, we must have $c_1 = \cdots = c_n = 0$. S spans V if every element in V is a linear combination of the elements of S. That is, for every $\vec{y} \in V$, there are coefficients $c_1, \ldots, c_n$ such that $\vec{y} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$.

(b) Let A be an $n \times n$ matrix, and let $\vec{x} \in \mathbb{R}^n$. Explain how you would determine whether $\vec{x} \in \operatorname{Col} A$ and whether $\vec{x} \in \operatorname{Nul} A$. (3 points)

Since $A\vec{y} = \vec{x}$ is consistent if and only if $\vec{x} \in \operatorname{Col} A$, you could row reduce the augmented matrix $[A\vec{x}]$ and $\vec{x} \in \operatorname{Col} A$ if and only if it corresponds to a consisted system. You could check if $\vec{x} \in \operatorname{Nul} A$ by multiplying A by $\vec{x}$, and seeing if you get $\vec{0}$.

(c) Define what it means for a vector space to be finite dimensional. (4 points)

A vector space is called finite dimensional if there is some (finite) set S such that $V = \operatorname{Span} S$. This is equivalent to V having a (finite) basis.

(d) Define what it means for a square matrix to be diagonalizable. (4 points)

A matrix A is called diagonal if it is similar to a diagonal matrix. That is, if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Problem 2. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 9 \\ 5 \\ 7 \end{bmatrix}$.

(a) Let $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Find an orthogonal basis for V (with respect to the dot product). (5 points)

We apply Gram-Schmidt.

$$\vec{x}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \qquad \vec{x}_2 = \vec{v}_2 - \text{Proj}_{\text{Span}\,\vec{x}_1} \vec{v}_2 = \vec{v}_2 - \frac{\vec{x}_1 \cdot \vec{v}_2}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix},$$

$$\vec{x}_3 = \vec{v}_3 - \text{Proj}_{\text{Span}\,\vec{x}_1, \vec{x}_2} \vec{v}_3 = \vec{v}_3 - \left(\frac{\vec{x}_1 \cdot \vec{v}_3}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1 + \frac{\vec{x}_2 \cdot \vec{v}_3}{\vec{x}_2 \cdot \vec{x}_2} \vec{x}_2 \right) = \begin{bmatrix} 1 \\ 9 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}.$$

Thus our orthogonal basis for V is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

(b) Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Find the vector in W closest to $\vec{v}_3$, and the distance between $\vec{v}_3$ and W. (5 points)

The closest vector in W to $\vec{v}_3$ is $\text{Proj}_W \vec{v}_3$. To calculate this, we need an orthogonal basis for W. Note that if we apply Gram-Schmidt to $\{\vec{v}_1, \vec{v}_2\}$ we would just get $\{\vec{x}_1, \vec{x}_2\}$ from above. That is, $\{\vec{x}_1, \vec{x}_2\}$ is an orthogonal basis for W. Thus

$$\text{Proj}_W \vec{v}_3 = \text{Proj}_{\text{Span}\,\vec{x}_1, \vec{x}_2} \vec{v}_3 = \begin{bmatrix} 4 \\ 8 \\ 6 \\ 5 \end{bmatrix}$$

was calculated in part (a), or it could be recalculated for fun from the formula using $\{\vec{x}_1, \vec{x}_3\}$. The distance from W to $\vec{v}_3$ is $\|\vec{v}_3 - \text{Proj}_W \vec{v}_3\|$. From our work in (a), this is just $\|\vec{x}_3\| = \sqrt{15}$.

Problem 3. Say whether the given statement is true or false. **If it is true, explain why. If it is false, provide a counterexample showing that it is false.** No points are given for true/false without correct justification. (3 points each)

(a) If A is diagonalizable, then all of its eigenvalues have multiplicity 1.

False. A counterexample is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) If the characteristic polynomial of A is $\lambda^2(\lambda-3)^4(\lambda+5)^6$, then $\dim \operatorname{Nul} A \leq 2$.

True. The characteristic polynomial tells us that 0 is an eigenvalue of A, with multiplicity 2. Then $\dim \operatorname{Nul} A$ is the dimension of the eigenspace for this eigenvalue, which must be less than or equal to the multiplicity.

(c) Orthogonal matrices are invertible.

True. An orthogonal matrix is one for which $U^T = U^{-1}$.

(d) If S_1 and S_2 are subsets of a vector space V such that S_1 is linearly dependent and S_2 does not span V, then S_2 has at least as many elements as S_1.

False; this is not always true. A counterexample is $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ and $S_2 = \left\{ \begin{bmatrix} 47 \\ 47 \end{bmatrix} \right\}$. Note: the statement *can* be true for certain choices of S_1 and S_2, e.g. $S_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ and $S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$.

(e) If S_1 and S_2 are subsets of a vector space V such that S_1 is linearly independent and S_2 spans V, then S_2 has at least as many elements as S_1.

True. S_1 has at most $\dim V$ elements, and S_2 has at least $\dim V$ elements. (We know V is finite dimensional because it admits a spanning set).

Problem 4. The sets $B = \left\{ \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ are bases for a common vector space V.

(a) Find $\underset{C \leftarrow B}{P}$ and $\underset{B \leftarrow C}{P}$. (Hint: solve for the elements of one basis in terms of the other.) (6 points)

Solving $\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ gives $c_1 = 1$ and $c_2 = 2$. Thus $[\vec{b}_1]_C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Similarly, solving $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ gives $c_1 = 1$ and $c_2 = 1$. Thus $[\vec{b}_2]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence

$$\underset{C \leftarrow B}{P} = [[\vec{b}_1]_C \quad [\vec{b}_2]_C] = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

and

$$\underset{B \leftarrow C}{P} = \underset{C \leftarrow B}{P}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

(b) If $T : V \to V$ is a linear transformation and $[T]_B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $[T]_C$. (4 points)

We have

$$[T]_C = \underset{C \leftarrow B}{P} [T]_B \underset{B \leftarrow C}{P} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 11 & -3 \end{bmatrix}.$$

Problem 5. Let A be a 5×5 matrix such that $A^2 = 0$ (the zero matrix). What are the possible value(s) of $\operatorname{Rank} A$? Narrow it down as much as possible, but you don't need to provide examples to prove that your possible values are, in fact, possible. (Hint: look at $\operatorname{Col} A$ and $\operatorname{Nul} A$) (10 points)

We first show that $\operatorname{Col} A \subseteq \operatorname{Nul} A$ (as in the practice problem). Suppose $\vec{y} \in \operatorname{Col} A$. Then there is some $\vec{x} \in \mathbb{R}^5$ such that $A\vec{x} = \vec{y}$. Hence $A\vec{y} = A^2\vec{x} = 0$, and so $\vec{y} \in \operatorname{Nul} A$. Since an arbitrary element of $\operatorname{Col} A$ must be in $\operatorname{Nul} A$, we have shown that $\operatorname{Col} A \subseteq \operatorname{Nul} A$.

We therefore know that $\operatorname{Rank} A = \dim \operatorname{Col} A \leq \dim \operatorname{Nul} A$. But the Rank-Nullity theorem says that $\operatorname{Rank} A + \dim \operatorname{Nul} A = 5$. The only possible way to satisfy these conditions is if $\operatorname{Rank} A$ is $0, 1$ or 2.

You didn't have to show that all of these values were possible, but we can do that now. Certainly $\operatorname{Rank} A = 0$ is possible, since $0^2 = 0$ and $\operatorname{Rank} 0 = 0$. For the other two:

$$\operatorname{Rank} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 1, \qquad \operatorname{Rank} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 2.$$

Problem 6. You are (still) a secret agent. Unfortunately, the matrix A you found on the previous exam was a decoy, and your organization actually needs information about the much larger matrix B. Via super secret methods, you are able to determine the following facts about B:

- B is $n \times n$, where $90 \leq n \leq 110$.
- 1 is an eigenvalue of B, and $\dim E_1 = 63$.
- -1 is an eigenvalue of B, and $\dim E_{-1} = 47$.

Somehow, your agency expects you to find B^2. Is this possible? If not, say as much as you can about B, B^2 and n. If it is possible, find B^2 and save the world for real this time. (10 points)

Since the dimensions of the eigenspaces cannot add up to more than n (by the diagonalization theorem at the end of the diagonalization section of the text), we must have $n = 110$. Thus $\dim E_1 + \dim E_{-1} = n$, and so B is diagonalizable. Thus there is an invertible matrix P and a diagonal matrix D, whose diagonal entries are ± 1, such that $B = PDP^{-1}$. Then

$$B^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1} = PI_{110}P^{-1} = PP^{-1} = I_{110}.$$