Name: ______________________________

Math 54, Summer 2009, Lecture 4
Quiz 2 Solution

(1) In (a) and (b), determine whether the given set is linearly independent or linearly dependent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 13 \\ 11 \\ 19 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 12 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \right\}$

(a) This set is linearly dependent, as there are more vectors than entries in the vectors.

(b) We have

$$\begin{bmatrix} 1 & 4 & -1 \\ 2 & 9 & 1 \\ 3 & 12 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix},$$

which is in echelon form. There is a pivot in every columns, so the columns of the original matrix are linearly independent.

(2) Is $\begin{bmatrix}1\\3\\7\end{bmatrix} \in \text{Span}\{\begin{bmatrix}11\\0\\0\end{bmatrix}, \begin{bmatrix}1\\4\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\end{bmatrix}\}$? Justify your answer.

There are two approaches to this problem. Approach 1: The matrix

$$\begin{bmatrix}11 & 1 & -2\\0 & 4 & 0\\0 & 1 & -1\end{bmatrix} \to \begin{bmatrix}11 & 1 & -2\\0 & 4 & 0\\0 & 0 & -1\end{bmatrix}$$

has a pivot in every row, so its columns span all of $\mathbb{R}^3$. In particular, $\begin{bmatrix}1\\3\\7\end{bmatrix}$ is in the span of the columns.

Approach 2: The question is equivalent to asking whether the following vector equation is consistent

$$x_1\begin{bmatrix}11\\0\\0\end{bmatrix} + x_2\begin{bmatrix}1\\4\\1\end{bmatrix} + x_3\begin{bmatrix}-2\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\3\\7\end{bmatrix}.$$

We can determine that by putting the following matrix in echelon form

$$\begin{bmatrix}11 & 1 & -2 & 1\\0 & 4 & 0 & 3\\0 & 1 & -1 & 7\end{bmatrix} \to \begin{bmatrix}11 & 1 & -2 & 1\\0 & 4 & 0 & 3\\0 & 0 & -1 & *\end{bmatrix}$$

where we haven't bothered to calculate the * entry. This echelon form has no row of the form $[\,0 \;\; 0 \;\; \cdots \;\; 0 \;\; b\,]$ with $b \neq 0$, so the corresponding system is consistent and the answer is yes.

(3) (a) Write the solution to

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

in parametric vector form.
(b) What is Span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$?

(a) First we find the general solution to this matrix equation by row reducing.

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 1 & -1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1/3 & 1/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & -1/3 & 1/3 \end{bmatrix}$$

This yields $x_1 = \frac{1}{3}x_3 + \frac{1}{3}$ and $x_2 = \frac{1}{3}x_3 + \frac{1}{3}$ with x_3 free. As a vector, this is

$$\begin{bmatrix} \frac{1}{3}x_3 + \frac{1}{3} \\ \frac{1}{3}x_3 + \frac{1}{3} \\ x_3 \end{bmatrix},$$

and in parametric vector form we have

$$\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \qquad (x_3 \in \mathbb{R}).$$

(b) While row reducing above, we saw that

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \end{bmatrix}.$$

Thus the matrix has a pivot in every row, and by a theorem this means that its columns span $\mathbb{R}^2$. Thus Span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2$.