Name: ______________________________

Math 54, Summer 2009, Lecture 4
Quiz 6 Solutions

(1) Determine if the statement is true or false, and **justify your answer in either case**. No points given without correct justification. Assume that V is finite-dimensional.

(a) If there exists a set $\{v_1, \dots, \vec{v}_p\}$ that spans V, then $\dim V \le p$.

True. The Spanning Set Theorem says that we may remove elements from a spanning set to obtain a basis, so a spanning set must contain at least as many elements as the dimension of the space.

(b) If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

True. A basis for V has p elements, and adding any element (e.g. $\vec{0}$) will not change the fact that the set spans.

(c) If $p \ge 2$ and $\dim V = p$, then every set of $p - 1$ vectors is linearly independent.

False. For example, $\dim \mathbb{R}^2 = 2$, but $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is a linearly dependent set. Another example is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ considered as a subset of $\mathbb{R}^3$.

(2) What is the dimension of Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \\ -1 \end{bmatrix} \right\}$? What is a basis for this vector space? (Hint: turn this into a question about the rank/column space of a matrix)

If we let V be the Span in the question, then

$$V = \mathrm{Col} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 5 \\ -1 & 0 & 0 & -1 \end{bmatrix}.$$

Row reducing this matrix, we find

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 5 \\ -1 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the column space is given by the pivot columns, so $\dim V = 3$ and a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$.

(3) Suppose that V and W are 3-dimensional vector spaces, and that $T : V \to W$ is a one-to-one linear transformation. Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V and prove that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is a basis for W.

Since T is one-to-one, and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, homework problem 4.3.32 says that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly independent. Since $\dim W = 3$, the two out of three theorem says that this set is a basis for W.

If you didn't remember or want to cite the homework problem, you can reprove that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly independent as follows. Suppose that $c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = \vec{0}$. By linearity, this means that $T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = \vec{0}$. Since T is one-to-one, this menas that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, this means that $c_1 = c_2 = c_3 = 0$. Thus we have shown that whenever a linear combination of $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is equal to $\vec{0}$, all coefficients must be 0, so the set is linearly independent. By the two out of three theorem, the set is a basis for W.