Name: ______________________________

Math 54, Summer 2009, Lecture 4
Quiz 8 Solution

(1) Let $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. If possible, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If it is impossible, say why. (Hint: the eigenvalues are 3 and -1) (4pts)

Since A is 2×2 and has two (distinct) eigenvalues, it will be diagonalizable. We need bases for the eigenspaces $\mathrm{Nul}(A - 3I)$ and $\mathrm{Nul}(A + I)$. To find the first, we solve $(A - 3I)\vec{x} = \vec{0}$:

$$\begin{bmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

That is, $2x_1 = x_2$. So $\mathrm{Nul}(A - 3I)$ can be described as $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for E_3 (Note: we could have solved for x_1 in terms of x_2, which would have given $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$, which would work just as well). Similarly, solving $(A + I)\vec{x} = \vec{0}$ yields that $E_{-1} = \mathrm{Span} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so our basis of eigenvectors is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$. Hence

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}.$$

(2) Say whether the given statement is true or false. If it is true, say why. If it is false, give a counterexample. (2pts)

(a) If A is a 3×3 matrix with eigenvalues 4, 5, and 7, then A is similar to $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

True. Since A has 3 distinct eigenvalues, it is diagonalizable by a theorem from class. Thus A is similar to the given matrix (and the other 3×3 diagonal matrices with 4, 5 and 7 on the diagonal).

(b) If A has 0 as a repeated eigenvalue (i.e. multiplicity > 1), then A is not diagonalizable.

False. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonal (and thus diagonalizable), and it has 0 as an eigenvalue with multiplicity two.

(3) Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$, and let W be the subspace spanned by the basis $B = \{e^x, xe^x\}$. Define $T : W \to W$ by $T(f) = f'$. Find $[T]_B$, and use this to determine $\operatorname{Ran} T$. (3pts)

Our formula says $[T]_B = [\ [T(e^x)]_B \quad [T(xe^x)]_B\ ]$. Since $\frac{d}{dx}e^x = e^x = 1(e^x) + 0(xe^x)$, we have $[T(e^x)]_B = [e^x]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Similarly, we have $[T(xe^x)]_B = [e^x + xe^x]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus $[T]_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Since $[T]_B$ has a pivot in every row, T is onto. Thus $\operatorname{Ran} T = W$.