Name: ______________________________

**Math 54, Summer 2009, Lecture 4**
**Quiz 9 Solution**

**(1)** Let $V = \mathbb{P}_2$ with inner product $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.

(a) If $p_1(t) = t^2$ and $p_2(t) = t$, show that $\{p_1, p_2\}$ is an orthogonal set. (4 points)

We just need to check that $\langle p_1, p_2\rangle = 0$. And indeed:

$$\langle p_1, p_2\rangle = p_1(-1)p_2(-1) + p_1(0)p_2(0) + p_1(1)p_2(1) = 1 * -1 + 0 * 0 + 1 * 1 = 0.$$

(b) Let $W = \text{Span}\,\{p_1, p_2\}$, and compute $\text{Proj}_W\, q$, where $q(t) = t + 1$.

Since we have an orthogonal basis for $W$, we have

$$\text{Proj}_W\, q = \frac{\langle q, p_1\rangle}{\langle p_1, p_1\rangle} p_1 + \frac{\langle q, p_2\rangle}{\langle p_2, p_2\rangle} p_2.$$

We can now compute $\langle q, p_1\rangle = 0 * 1 + 1 * 0 + 2 * 1 = 2$, $\langle p_1, p_1\rangle = 1 * 1 + 0 * 0 + 1 * 1 = 2$, $\langle q, p_2\rangle = 0 * -1 + 1 * 0 + 2 * 1 = 2$ and $\langle p_2, p_2\rangle = -1 * -1 + 0 * 0 + 1 * 1 = 2$. Thus $\text{Proj}_W\, q = \frac{2}{2}t^2 + \frac{2}{2}t = t^2 + t$.

**(2)** True or false? If it is true, say why. If it is false, provide a counterexample (3 points)

(a) If $V = \mathbb{R}^4$, an $\{\vec{u}_1, \vec{u}_2\}$ and $\{\vec{u}_3, \vec{u}_4\}$ are orthogonal subsets of $V$, then so is $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$.

False. Consider $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$

(b) If $U$ is an orthogonal matrix and you rearrange the columns of $U$, then the resulting matrix is still orthogonal.

True. A matrix is orthogonal if and only if the columns are an orthonormal basis for $\mathbb{R}^n$. If we reorder an orthonormal basis for $R^n$, then it will still be an orthonormal basis.

**(3)** Suppose that $A$ is a square matrix, and that $\vec{x}_0$ is an eigenvector of $A$ with eigenvalue $\lambda$. Suppose also that $A\vec{x}_0$ is orthogonal to $\vec{x}_0$. What does this tell you about $\lambda$? What does this tell you about $A$? (2 points)

We have $A\vec{x}_0 \cdot \vec{x}_0 = 0$ and $A\vec{x}_0 = \lambda\vec{x}_0$. Combining yields $(\lambda\vec{x}_0) \cdot \vec{x}_0 = 0$. Pulling the scalar out of the inner product, we get $\lambda(\vec{x}_0 \cdot \vec{x}_0) = 0$. Now, $\vec{x} \cdot \vec{x} = 0$ only when $\vec{x} = \vec{0}$, but by definition eigenvectors cannot be $\vec{0}$. Thus $\vec{x}_0 \cdot \vec{x}_0 \neq 0$, so we must conclude that $\lambda = 0$. This means that $A$ is not invertible.