Name: ______________________________

Math 54, Summer 2009, Lecture 4
Quiz 11 Solutions

(1) Write the following ODE both as a system of first order linear equations, and as a matrix system in normal form

$$y''' + (\cos t)y'' - 16y' + 5y = e^t.$$

An equivalent system of equations in the functions y, x_1 and x_2 is

$$\begin{aligned} y' &= x_1 \\ x_1' &= x_2 \\ x_2' &= -(\cos t)x_2 + 16x_1 - 5y + e^t. \end{aligned}$$

As a matrix system, this turns into

$$\begin{bmatrix} y \\ x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 16 & -\cos t \end{bmatrix} \begin{bmatrix} y \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix}.$$

(2) Find all functions $y(t)$ such that $y'' - y' - 2y = 3e^{-t}$.

The auxiliary equation of the homogeneous equation is $0 = r^2 - r - 2 = (r-2)(r+1)$. Thus the general solution to the homogenous equation is $c_1e^{-t} + c_2e^{2t}$. We now guess $y_p = Ate^{-t}$ (with an extra factor of t because -1 is a single root of the auxiliary equation). This gives

$$\begin{aligned} y_p' &= Ae^{-t} - Cte^{-t}, \\ y_p'' &= -2Ae^{-t} + Ate^{-t}. \end{aligned}$$

Thus we set

$$3e^{-t} = y_p'' - y_p' - 2y_p = -3Ae^{-t}$$

and get $A = -1$. Thus the general solution to the ODE is $-te^{-t} + c_1e^{-t} + c_2e^{2t}$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary.

(3) Prove that $\sin(t+a) = \cos a \sin t + \sin a \cos t$ for all numbers t and a. Hint: think of a as arbitrary, but fixed, and think of t as a variable. Consider the IVP

$$y'' + y = 0, \quad y(0) = \diamond, \quad y'(0) = \circ,$$

for judicious choices of $\diamond$ and $\circ$.

Consider the initial value problem

$$y'' + y = 0, \quad y(0) = \sin(a), \quad y'(0) = \cos(a).$$

One can check that both $y_1 = \sin(t+a)$ and $y_2 = \cos a \sin t + \sin a \cos t$ are solutions to this IVP. However, a theorem from class says that IVPs such as this one have unique solutions on $\mathbb{R}$, so we must have $\sin(t+a) = \cos a \sin t + \sin a \cos t$ for all t. Since a was arbitrary, this holds for all a and all t.