Name: ________________________________

Math 54, Summer 2009, Lecture 4
Quiz 12 Solutions

(1) Solve the initial value problem $\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, where $A = \begin{bmatrix} -1 & -9 \\ 0 & 2 \end{bmatrix}$. (4 points)

Since A is upper triangular, its eigenvalues are the numbers on the diagonal, -1 and 2. We row reduce to find bases for the eigenspaces

$$[A - 2I \quad \vec{0}] = \begin{bmatrix} -3 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which corresponds to the equation $x_1 = -3x_2$, so that $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a basis for the eigenspace. Similarly we row reduce $A + I$ and get

$$\begin{bmatrix} 0 & -9 & 0 \\ 0 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $x_2 = 0$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a basis for the eigenspace. So the general solution to the system of ODEs is

$$c_1 e^{2t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Plugging in the initial condition we get $-3c_1 + c_2 = 4$ and $c_1 = -2$. This gives $c_2 = -2$, so the solution to the IVP is

$$-2e^{2t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} - 2e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6e^{2t} - 2e^{-t} \\ -2e^{2t} \end{bmatrix}.$$

(2) Let $A = \begin{bmatrix} -1 & -9 \\ 0 & 2 \end{bmatrix}$ as in (1).

(a) Find e^{At}. (3 points)

Option 1: From (1), we have that $X(t) = \begin{bmatrix} -3e^{2t} & e^{-t} \\ e^{2t} & 0 \end{bmatrix}$ is a fundamental matrix for $\vec{x}\,' = A\vec{x}$. Thus

$$e^{At} = X(t)X(0)^{-1} = \begin{bmatrix} -3e^{2t} & e^{-t} \\ e^{2t} & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} e^{-t} & 3(e^{-t} - e^{2t}) \\ 0 & e^{2t} \end{bmatrix}.$$

Option 2: Based on the data from (1), we have

$$A = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

Thus

$$e^{At} = \operatorname{Exp}\left(\begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2t & 0 \\ 0 & -t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right) = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \operatorname{Exp}\left(\begin{bmatrix} 2t & 0 \\ 0 & -t \end{bmatrix} \right) \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} e^{-t} & 3(e^{-t} - e^{2t}) \\ 0 & e^{2t} \end{bmatrix}.$$

(b) Find a fundamental matrix $X(t)$ for $\vec{x}\,' = A\vec{x}$ such that $X(0) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$. Hint: Look for a matrix of the form $e^{At}P$. (2 points)

Since e^{At} is a fundamental matrix, so is $e^{At}P$ for any invertible matrix P. Since $e^{At} = I$, we need $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = e^{A0}P = P$. Thus our fundamental matrix is

$$\begin{bmatrix} e^{-t} & 3(e^{-t} - e^{2t}) \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3(e^{-t} - e^{2t}) & 2e^{-t} \\ e^{2t} & 0 \end{bmatrix}.$$