Name: ______________________________

Math 54, Summer 2009, Lecture 4
"Quiz 13" Solutions

(1) Consider the PDE $u_{xx} + u_x - u_t = 0$. Derive a pair of ODEs that $X(x)$ and $T(t)$ would have to satisfy for $u(x,t) = X(x)T(t)$ to satisfy this PDE. (3 points)

Plugging in $u(x,t) = X(x)T(t)$ gives $X''T + X'T - XT' = 0$, or

$$\frac{X'' + X'}{X} = \frac{T'}{T}.$$

Since the left hand side is a function of x and the right hand side is a function of t, they must both be equal to a common constant K. Thus we get

$$X'' + X' - KX = 0$$

and

$$T' - KT = 0.$$

(2) (a) Let $f(x) = (1-x)(e^x - 1)$. Set up the Fourier series for f on $[-\pi, \pi]$, and the Fourier sine and cosine series for f on $[0, \pi]$. By "set up", I mean that you do not need to evaluate any integrals, just write them down. (4 points)

The Fourier series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1-x)(e^x - 1)\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1-x)(e^x - 1)\sin(nx)dx.$$

The Fourier cosine series in $\frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos nx$ where

$$c_n = \frac{2}{\pi}\int_{0}^{\pi} (1-x)(e^x - 1)\cos(nx)dx.$$

The Fourier sine series is $\sum_{n=1}^{\infty} d_n \sin nx$ where

$$d_n = \frac{2}{\pi}\int_{0}^{\pi} (1-x)(e^x - 1)\sin(nx)dx.$$

(b) Write down a formal solution to the heat problem $\begin{cases} u_t = u_{xx} & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0 & t > 0, \\ u(x,0) = f(x) & 0 < x < \pi, \end{cases}$
where $f(x)$ is as in (a). Again, do not evaluate any of the integrals. (2 points)

A formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} d_n u_n(x,t) = \sum_{n=1}^{\infty} d_n e^{-n^2 t}\sin(nx),$$

where d_n is as above.