Math 54, Summer 2009, Lecture 4
Worksheet 2: Lay 4.1-4.2
Solutions

(1) Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$. Describe how you would solve the following questions, but don't actually do the computation.

(a) Is $\begin{bmatrix} 5 \\ 5 \end{bmatrix} \in \operatorname{Col} A$? Same as determining if $A\vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ is consistent by row reducing the appropriate augmented matrix.

(b) Is $\begin{bmatrix} 5 \\ 5 \end{bmatrix} \in \operatorname{Nul} A$? No. $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^3$ in this case, not $\mathbb{R}^2$.

(c) Is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \operatorname{Col} A$? No. $\operatorname{Col} A$ is a subspace of $\mathbb{R}^2$, not $\mathbb{R}^3$.

(d) Is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \operatorname{Nul} A$? Check if $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$ by multiplying it out.

(e) Find something non-zero in $\operatorname{Nul} A$: The elements of $\operatorname{Nul} A$ are solutions to the equation $A\vec{x} = \vec{0}$, so row reduce the augmented matrix, write out the general solution, and choose a non-zero value for at least one free variable.

(f) Find something non-zero in $\operatorname{Col} A$: If A is not the zero matrix, just pick your favorite non-zero column. If A is the zero matrix, $\operatorname{Col} A = \{\vec{0}\}$, so this is impossible.

(2) Let $W = C[a,b]$, the vector space of continuous functions $f : [a,b] \to \mathbb{R}$. Let $V = C^1[a,b]$, the vector space of differentiable functions $f : [a,b] \to \mathbb{R}$ whose derivative is continuous. Define $D : V \to W$ by $D(f) = f'$.

(a) Explan why it makes sense for $D : V \to W$, and show that D is a linear transformation.

(b) What is $\operatorname{Ker} D$?

(c) Bonus: show that D is onto.

(a) If $f \in V$, then it has a continuous derivative, so $f' \in W$. Thus it makes sense to have $D : V \to W$. It is a basic fact from calculus that $(f+g)' = f' + g'$ and that $(cf)' = c(f')$, which shows that D is linear.

(b) $f \in \operatorname{Ker} D \iff D(f) = \vec{0} \iff f'(x) = 0$ for all x. From calculus, this holds exactly when f is a constant function, so the kernel of D consists of all the constant functions.

(c) We need to show that for every $g \in W$, there is some $f \in V$ such that $D(f) = g$. So fix a continuous function g, and define

$$f(x) = \int_0^x g(t)dt.$$

By the fundamental theorem of calculus, f is differentiable, and $f' = g$. This shows simultaneously that $f \in V$ (because its derivative, g, is continuous), and that $D(f) = g$.

(3) Let $V = M_{2\times 3}$, the vector space of 2×3 matrices with real entries. Assume that A is a 3×3 matrix, and that A is invertible. Define $T : V \to W$ by $T(B) = BA$.

(a) What should W be?

(b) Show that T is a linear transformation.

(c) What is $\operatorname{Ker} T$? What is $\operatorname{Ran} T$?

(a) T takes a 2×3 matrix B as input, and multiplies it on the right by a 3×3 matrix A. The result is the 2×3 matrix BA, so an appropriate codomain is $M_{2\times 3}$.

(b) If $B, C \in M_{2\times 3}$, then basic matrix multiplication rules say that

$$T(B + C) = (B + C)A = BA + CA = T(B) + T(C)$$

and

$$T(cB) = cBA = cT(B).$$

So T is linear.

(c) Suppose $T(B) = 0$, the zero matrix. Then $BA = 0$. Multiplying on the left by A^{-1} yields that $B = 0$, and so the zero matrix is the only element of $M_{2\times 3}$ that is sent to zero. Thus $\operatorname{Ker} T = \{0\}$ and T is 1-1.

We now show that T is onto (and so $\operatorname{Ran} T = M_{2\times 3}$). Fix an arbitrary $C \in M_{2\times 3}$, and we will now find some $B \in M_{2\times 3}$ such that $T(B) = C$ (showing that C is in the range of T). If we set $B = CA^{-1}$, then $T(B) = CA^{-1}A = C$, as desired. How did we choose B? We wanted $T(B) = C$, or in other words $BA = C$. Solving for B by multiplying on both sides by A^{-1} gives $B = CA^{-1}$.