Math 54, Summer 2009, Lecture 4
Worksheet 3: Lay 4.3

Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -2 \\ 3 & 1 & 4 \end{bmatrix}$. In a group, use the following ideas to find a basis for $\mathrm{Col}\, A$.

a) Explain why the pivot columns of an echelon form of A are linearly independent, and why no bigger set of columns containing the pivot columns is linearly independent (Hint: think of about pivots in the matrix formed by the set of columns).

b) Explain why echelon forms of A can be factored PA, where P is invertible.

c) Show that for any $\vec{x} \in \mathbb{R}^3$, we have $A\vec{x} = \vec{0}$ if and only if $PA\vec{x} = \vec{0}$, and use this to show that a set of columns in A is linearly independent if and only if the corresponding columns in ecehlon form are linearly independent.

d) What is a basis for $\mathrm{Col}\, A$?

a) The matrix formed from the pivot columns has a pivot in every column, so the pivot columns are linearly independent. However, the matrix formed from any bigger set of columns containing the pivot columns would not have a pivot in every column, and would therefore be linearly dependent.

b) Echelon forms are of the form $E_1 \cdots E_m A$, where E_j is elementary. Elementary matricecs are invertible, so $P = E_1 \cdots E_m$ is invertible.

c) If $A\vec{x} = \vec{0}$, then multiplying both sides by P gives $PA\vec{x} = \vec{0}$. If $PA\vec{x} = \vec{0}$, then multiplying both sides by P^{-1} gives $A\vec{x} = \vec{0}$. Now observe that columns 1, 3 and 5 (say) of A are linearly dependent if and only if $A\vec{x} = \vec{0}$ has a nontrivial solution where all the entries besides x_1, x_3 and x_5 are 0. Thus a set of columns in A is linearly independent if and only if the corresponding columns in echelon form are linearly independent. Thus the pivot columns of A are a maximal linearly independent set of columns, and therefore a basis for $\operatorname{Col} A$.

e) Row reducing shows that A has pivots in the first two columns, so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\operatorname{Col} A$.