The Classification of Extremal Vertex Operator Algebras of Rank 2

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Abstract

In this paper we provide an overview of vector-valued modular forms, an algebraic structure that appears in number theory. We describe how they can be used to study vertex operator algebras, which are in turn algebraic representations of certain conformal field theories. Through this connection we prove that the number of character vectors for extremal vertex operator algebras with representation categories $\mathcal{C}$ satisfying $\text{rank}(\mathcal{C}) = 2$ is finite. Moreover, we provide a list of all possible candidates for these character vectors.

1 Introduction

One of the few classes of rigorously defined quantum field theories are conformal field theories (CFTs). A subset of these, unitary chiral conformal field theories (χCFT), are associated to a mathematical structure called a vertex operator algebra (VOA), which encodes the chiral algebra of the χCFT. In addition, nice enough χCFTs give rise to unitary modular tensor categorys (UMCs) which can be constructed from their associated VOAs. The question of how to recreate a VOA from a UMC and how much additional information in necessary to do so is still an active area of research [EG18].

Rather than tackle this problem directly, our goal is simply to use UMCs to classify the character vectors of VOAs, which were shown by [Zhu96] to be vector-valued modular forms. We restrict our attention to so-called extremal

VOAs, who's well-behavedness allow us to use the results of [TW17, Thm. 3.1]. By further restricting ourselves to UMCs $\mathcal{C}$ such that $\text{rank}(\mathcal{C}) = 2$, this theorem shows that the character vector of an associated VOA is uniquely determined by a single parameter, the central charge c. Our results show that only a finite number of pairs $(\mathcal{C}, c)$, called genera, produce viable candidate character vectors. We further show that the list given in [TW17, Sec. 3.2] of some of these candidate character vectors is in fact complete.

We first give a brief outline of some relevant elementary algebra in section 2. In section 3 we provide some background on modular froms. Then in section 4 we define vector-valued modular forms (VVMFs) and describe the method given in [BG07] of constructing the fundamental matrix. In section 5, we give an overview of extremal VOAs and how each induces a VVMF. In section 6 we prove a theorem about VVMFs which allows one to completely classify the character vectors of extremal VOAs $\mathcal{V}$ with associated UMC $\text{Rep}(\mathcal{V})$ that satisfy $\text{rank}(\text{Rep}(\mathcal{V})) = 2$. In section 7 we prove the complete list of these character vectors.

2 Some Basic Algebra

A *group* is a pair $(G, \cdot)$, where G is a set and $\cdot$ is a binary operation on the set, satisfying the following properties:

Closure: For all $a, b \in G$, $a \cdot b \in G$.

Associativity: For all $a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Identity: There exists an element $1 \in G$ that satisfies $1 \cdot a = a \cdot 1 = a$ for all $a \in G$.

Inverse: There exists an element $a^{-1} \in G$ such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$ for all $a \in G$.

An *abelian group* is a group which satisfies the following additional property:

Commutativity: For all $a, b \in G$, $a \cdot b = b \cdot a$.

By abuse of notation, from here on we will simply state that G is a group when the binary operation $\cdot$ is clear from context. We will similarly use the underlying set as a shorthand all of the algebraic structures mentioned in this paper when they are clear from context. In addition, we will often drop the $\cdot$, so $a \cdot b$ becomes ab.

A *subgroup* H of a group G is a subset of G that follows all of the group axioms when the binary operation is restricted to the set H. Given a subgroup H of G, and an element $a \in G$, a *left coset* is the set

$$aH = \{ah | h \in H\}.$$

The *right coset* Ha is defined analogously. A *normal subgroup* N of G is a subgroup such that for all $a \in G$, we have that $aN = Na$. Note that for an abelian group, all subgroups are normal. Given a group G and a normal subgroup N, the *quotient group* G/N is the set of left cosets of N where the binary operation is defined by

$$(aN)(bN) = (ab)N$$

for all $a, b \in G$.

A *group action* $\cdot$ of a group G on a set X is a mapping $\cdot : G \times X \to X$ that satisfies the following properties:

Compatibility $(ab) \cdot x = a \cdot (b \cdot x)$ for all $a, b \in G$ and $x \in X$.

Identity $1 \cdot x = x$ for all $x \in X$.

The group G is said to act on X. Because of the compatibility condition and the existence of inverse elements in G, the group action associates to each element of G a bijective map $X \to X$.

A *ring* is a triple $(R, +, \cdot)$, where $(R, +)$ is an abelian group (with the identity element denoted 0 instead of 1) and which satisfies the following additional properties:

Multiplicative Associativity: For all $a, b, c \in R$, $(ab)c = a(bc)$.

Multiplicative Identity: There exists an element $1 \in R$ that satisfies $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

Left Distributivity: $a(b + c) = ab + ac$ for all $a, b, c \in R$.

Right Distributivity: $(a + b)c = ac + bc$ for all $a, b, c \in R$.

The operations $+, \cdot$ are called addition and multiplication, respectively. A *commutative ring* is a ring in which multiplication is also commutative. A *field* is a commutative ring that contains multiplicative inverses for all non-zero elements.

A *polynomial ring* $R[x]$ in the indeterminate x over the ring R is the set of polynomials in x with coefficients in R, i.e. elements of of the form

$$a(x) = \sum_{i=0}^{n} a_i x^i,$$

where each $a_i \in R$ and $n \in \mathbb{N}$. Note that, as usual, $x^0 = 1$. The largest $i \in \{1, \ldots, n\}$ such that $a_i \neq 0$ is called the *degree* of $a(x)$. Let $b(x) \in R[x]$, and without loss of generality assume $\deg(b(x)) = m \geq n = \deg(a(x))$. Then addition in $R[x]$ is defined by

$$a(x) + b(x) = \sum_{i=0}^{m} (a_i + b_i) x^i$$

where if $n < i \leq m$ we set $a_i = 0$. Multiplication in $R[x]$ is defined by

$$a(x)b(x) = \sum_{i=0}^{m+n} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i$$

These operations make $R[x]$ into a commutative ring.

For any commutative ring R the *general linear group* $\mathrm{GL}(n, R)$ is the group of invertible $n \times n$ matrices under matrix multiplication whose entries are in R, where $n \in \mathbb{N}$. The *special linear group* $\mathrm{SL}(n, R)$ is the subgroup of $\mathrm{GL}(n, R)$ that contains only those matrices with determinant 1. The *group of scalar transformations* $\mathrm{Z}(n, R)$ is the subgroup of $\mathrm{GL}(n, R)$ whose elements are of the form λI_n for some $\lambda \in R - \{0\}$, where I_n is the identity matrix of size n. We define $\mathrm{SZ}(n, R)$ to be the subgroup of $\mathrm{Z}(n, R)$ of elements with unit determinant. The *projective special linear group* $\mathrm{PSL}(n, R)$ is the quotient group $\mathrm{SL}(n, R)/\mathrm{SZ}(n, R)$.

A *left module* M over a ring R is an abelian group $(M, +)$ along with a mapping $\cdot : R \times M \to M$ that satisfies the following properites:

Left Distributivity: $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in R$ and $x, y \in M$.

Right Distributivity: $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in R$ and $x \in M$.

Compatibility: $(ab) \cdot x = a \cdot (b \cdot x)$ for all $a, b \in R$ and $x \in M$.

Identity: $1 \cdot x = x$ for all $x \in M$.

Again, it is customary to simply drop the mapping $\cdot$. *Right modules* over R are defined similarly. If R is a commutative ring, then left and right R-modules are equivalent and are simply called *R-modules.* Note that R is itself and R-module.

A *free module* is a module that admits a basis. If F is a field, then an F-module is called a *vector space* over F. A *submodule* N of an R-module M is a subgroup of M such that for all $a \in R$ and $x \in N$, $ax \in N$. An *irreducible module* is a module that has no proper, non-trivial submodules.

For any algebraic structure, a *homomorphism* is a mapping that preserves that structure. For example, if G, H are groups, a group homomorphism $\phi : G \to H$ must satisfy $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$. An *isomorphism* is a homomorphism that is also a bijection. An *isomorphism class* is a collection of mathematical objects that are isomorphic. An *automorphism* is an isomorphism from a mathematical object to itself. In general, the set of automorphism for a given mathematical object create a group, where the group operation is composition of maps.

For a module M over a commutative ring R we define the *dual module* $M*$ of M to be the R-module of module homomorphisms from M to R. If M is actually a vector space V over a field F, then the dual module $V*$ is called the *dual vector space* of V.

For a vector space V over a field F we define the *general linear group* $\mathrm{GL}(V)$ to be the automorphism group of V. A *representation* of a group G on V is a group homomorphism $\phi : G \to \mathrm{GL}(V)$. If n is the dimension of V, then $\mathrm{GL}(V)$ and $\mathrm{GL}(n, F)$ are isomorphic as groups, so we can equivalently think of a representation of G on V as a group homomorphism $\phi' : G \to \mathrm{GL}(n, F)$. We define $\dim(\phi) = \dim(V)$. A *subrepresentation* is the restriction of ϕ to a subspace $W \subset V$ such that for all $g \in G$ and for all $w \in W$, $\phi(g)w \in W$. An *irreducible representation* is a representation with no proper, non-trivial subrepresentations.

A matrix $U \in \mathrm{GL}(n, \mathbb{C})$ is *unitary* if U is invertible and $U^* = U^{-1}$, where U^* is the conjugate transpose of U. The *unitary group* $\mathrm{U}(n)$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of all $n \times n$ unitary matrices. A *unitary representation* is a representation whose codomain is $\mathrm{U}(n)$.

3 Modular Forms

A *linear fractional transformation* is a mapping $f : \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

We define the *modular group* to be $\mathrm{PSL}(2, \mathbb{Z})$. Note that for any $\lambda \in \mathbb{Z}$

$$\det(\lambda I_2) = \lambda^2 \det(I_2) = \lambda^2,$$

so $\mathrm{SZ}(2, \mathbb{Z}) = \{\pm I_2\}$. The modular group is generated by the cosets

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \{\pm I_2\},$$

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \{\pm I_2\}.$$

Proposition 3.1. *For any coset $\gamma \in PSL(2, \mathbb{Z})$ we choose a representative*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the action of γ on the upper half-plane $\mathbb{H}$ by linear fraction transformation given by

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

where $\tau \in \mathbb{H}$, is independent of the choice of representative of γ. Moreover, this is a group action.

Proof. First, note that

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \{\pm I_n\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$$

so there are only two elements in the coset γ. We immediately see that

$$\frac{(-a)\tau + (-b)}{(-c)\tau + (-d)} = \frac{a\tau + b}{c\tau + d}.$$

This then shows that the action of γ can be equally calculated from either of its representatives. Thus as a shorthand we will write elements of the modular group as single representatives in the future, rather than as whole cosets.

Next, we see that

$$T \cdot \tau = \frac{1\tau + 1}{0\tau + 1} = \tau + 1$$
$$S \cdot \tau = \frac{0\tau - 1}{1\tau + 0} = \frac{-1}{\tau}.$$

Since T and S generate $\mathrm{PSL}(2, \mathbb{Z})$ and these transformations then preserve $\mathbb{H}$, $\mathbb{H}$ is closed under all $\gamma \in \mathrm{PSL}(2, \mathbb{Z})$.

Lastly, we calculate

$$\begin{aligned}\gamma' \cdot (\gamma \cdot \tau) &= \gamma \cdot \frac{a\tau + b}{c\tau + d} \\ &= \frac{a' \frac{a\tau+b}{c\tau+d} + b'}{c' \frac{a\tau+b}{c\tau+d} + d'} \\ &= \frac{(a'a + b'c)\tau + (a'b + b'd)}{(c'a + d'c)\tau + (c'b + d'd)} \\ &= (\gamma'\gamma) \cdot \tau.\end{aligned}$$

Thus the action is compatible with the group operation. □

A function $f : D \subset \mathbb{C} \to \mathbb{C}$, where D is open, is called *holomorphic* on D if it is differentiable in a neighborhood of every point in D. This condition in fact implies that f is infinitely differentiable. We call f *meromorphic* if f is holomorphic except at at set of isolated points.

A *modular form* of weight k for the modular group is a function $f : \mathbb{H} \to \mathbb{C}$ that is holomorphic in the upperhalf plane $\mathbb{H}$ and satisfies

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau),$$

where $\tau \in \mathbb{H}$ and

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2, \mathbb{Z}).$$

Modular functions are defined similarly to modular forms of weight 0, except the holomorphicity requirement is relaxed to allow functions that are meromorphic on $\mathbb{H}$ so long as their only pole is of finite order and is located at the point $i\infty$, called the *cusp*. That is to say, if we define $q = e^{2\pi i\tau}$ for all $\tau \in \mathbb{H}$ and let f be a modular function, then the Laurent expansion of f takes the form

$$f(\tau) = \sum_{n=-m}^{\infty} a_n q^n$$

for some $m \in \mathbb{N}$. This is the so-called q-expansion of f.

Some specific modular forms that will be important to us are the *Eisenstein series*

$$E_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{1}{(m+n\tau)^k},$$

a modular form of weight k, and the *modular discriminant*

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

a modular form of weight 12. Also of use to us will be the *Klein J-invariant*

$$J(\tau) = q^{-1} + \sum_{n=1}^{\infty} j(n)q^n = q^{-1} + 196884q + \cdots,$$

which is the unique modular function of weight 0 that has only a simple pole at the cusp satisfying the relations $J(i) = 984$ and $J(e^{2\pi i/3}) = -744$.

For the interested reader, further background on the modular group and modular forms can be found in [Apo90].

4 Vector-Valued Modular Forms

Let ρ be an irreducible unitary representation of $\mathrm{PSL}(2,\mathbb{Z})$ of dimension d. We define a *vector-valued modular form* (VVMF) to be a holomorphic function $\mathbb{X} : \mathbb{H} \to \mathbb{C}^d$ that satisfies

$$\mathbb{X}(\gamma\cdot\tau) = \rho(\gamma)\mathbb{X}(\tau)$$

for all $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$ and whose component functions $\mathbb{X}_i$ have finite order poles at the cusp. We define $\mathcal{M}(\rho)$ to be the set of VVMFs associated to ρ.

Fix some $\Lambda \in U(d)$, called an *exponent matrix* of ρ, satisfying

$$e^{2\pi i\Lambda} = \sum_{n=0}^{\infty} \frac{(2\pi i)^n}{n!}\Lambda^n = \rho(T).$$

This is guaranteed to exist because $\rho(T)$ is unitary and thus invertible.

Lemma 4.1. *$e^{-2\pi i\tau\Lambda}\mathbb{X}(\tau)$ is invariant under integer translations in τ.*

Proof.

$$\begin{aligned} e^{-2\pi i(\tau+1)\Lambda}\mathbb{X}(\tau+1) &= e^{-2\pi i\tau\Lambda}e^{-2\pi i\Lambda}\mathbb{X}(T\cdot\tau) \\ &= e^{-2\pi i\tau\Lambda}e^{-2\pi i\tau}\rho(T)\mathbb{X}(\tau) \\ &= e^{-2\pi i\tau\Lambda}\mathbb{X}(\tau), \end{aligned}$$

□

Again, we define $q = e^{2\pi i\tau}$. Then this lemma proves that the function $q^{-\Lambda}\mathbb{X}(\log(q)/2\pi i)$ is a well-defined, single-valued function of q. In the rest of this paper by abuse of notation we will simply write $f(q)$ for the function $f(\log(q)/2\pi i)$ and will switch freely between notations $f(q)$ and $f(\tau)$ depending on what is most convenient.

We Fourier expand this expression as

$$q^{-\Lambda}\mathbb{X}(q) = \sum_{n\in\mathbb{Z}} \mathbb{X}[n]q^n$$

where $\mathbb{X}[n]$ are some coefficient vectors. We define the *principle part map* $\mathcal{P}_\Lambda$ to be the finite sum

$$\mathcal{P}_\Lambda\mathbb{X}(q) = \sum_{n<0} \mathbb{X}[n]q^n.$$

We call Λ *bijective* if $\mathcal{P}_\Lambda$ is an isomorphism of $\mathcal{M}(\rho)$. It can be shown [Gan14, Thm. 3.2] that bijective Λ exist for all ρ, that a bijective Λ must satisfy

$$\mathrm{Tr}(\Lambda) = \frac{5d}{12} + \frac{1}{4}\mathrm{Tr}(\rho(S)) + \frac{2}{3\sqrt{3}}\mathrm{Re}(e^{-i\pi/6}\mathrm{Tr}(\rho(ST^{-1})), \tag{1}$$

and that this condition is sufficient to ensure that Λ is bijective for $d < 6$ [Gan14, Thm. 4.1].

Given bijective Λ, we define the *canonical basis vector* $\mathbb{X}^{(\xi;n)} \in \mathcal{M}(\rho)$ to be the unique VVMF satisfying

$$[\mathcal{P}_\Lambda \mathbb{X}^{(\xi;n)}]_\eta = q^{-n}\delta_{\xi\eta},$$

where δ is the Kronecker delta. Multiplication by powers of J takes $\mathcal{M}(\rho)$ to itself, implying that $\mathcal{M}(\rho)$ is a $\mathbb{C}[J]$-module. It was shown in [BG06] that the canonical basis vectors satisfy the recursion relation

$$\mathbb{X}^{(\xi;n+1)} = J(\tau)\mathbb{X}^{(\xi;n)} - \sum_{m=1}^{n-1} j(n)\mathbb{X}^{(\xi;n-m)} - \sum_{\eta=1}^{d} \chi_\eta^{(\xi;n)}\mathbb{X}^{(\eta;1)}$$

where

$$\chi_\eta^{(\xi;n)} = \mathbb{X}^{(\xi;\eta)}[0]_\eta$$

is the constant part of $\mathbb{X}^{(\xi;n)}$. This allows us to express any canonical basis vector $\mathbb{X}^{(\xi;n)}$ as a linear combination

$$\mathbb{X}^{(\xi;n)} = \sum_{\eta=1}^{d} a_\eta(J)\mathbb{X}^{(\eta;1)},$$

where each $a_\eta(J) \in \mathbb{C}[J]$. This implies that the set of canonical basis vectors with simple poles $\{\mathbb{X}^{(\eta;1)}\}$ span $\mathcal{M}(\rho)$. In fact, it was additionally shown in [BG07] that this set forms a basis for $\mathcal{M}(\rho)$ as a free $\mathbb{C}[J]$-module, implying that $\mathcal{M}(\rho)$ has rank d.

We define the *fundamental matrix*

$$\Xi_{\xi\eta} = [\mathbb{X}^{(\eta;1)}]_\xi$$

and the *characteristic matrix*

$$\chi_{\xi\eta} = [\mathbb{X}^{(\eta;1)}[0]]_\xi$$

so χ is the constant part of Ξ. It was shown in [BG07] that one can solve for χ uniquely, up to the conjugation by a matrix commuting with Λ. It was further shown in [BG07, Sec. 2] that Ξ satisfies the recurrence relation

$$\frac{1}{2\pi i}\frac{d\Xi(\tau)}{d\tau} = \Xi(\tau)\mathcal{D}(\tau) \tag{2}$$

where

$$\mathcal{D}(\tau) = \frac{\Delta(\tau)}{E_{10}(\tau)}\Big((J(\tau) - 240)(\Lambda - I_d) + \chi + [\Lambda, \chi]\Big),$$

and $[\cdot\,,\cdot]$ is the commutator. This then allows one to find the rest of the coefficients of Ξ, again up to the conjugation by a matrix commuting with Λ. By construction Ξ must satisfy the relation

$$\Xi(\gamma \cdot \tau) = \rho(\gamma)\Xi(\tau),$$

so then solving this equation explicitly for fixed choices of γ and τ removes the remaining ambiguity.

5 Vertex Operator Algebras

Vertex operator algebras (VOAs) are algebraic structures that correspond to the chiral algebras of χCFTs [Kac98]. In this section, rather than delve into their structure, we will simply give a brief sketch of some important properties of VOAs and of how a class of VOAs can be associated to vector-valued modular forms. Throughout this paper, we assume certain niceness properties of VOAs that we will not define. Specifically, we assume that they are of rational, C_2-cofinite, and CFT type.

Let $\mathcal{V}$ be a nice VOA. Then $\mathcal{V}$ has finitely many isomorphism classes of irreducible modules, called *$\mathcal{V}$-modules*. From each of these isomorphism classes we choose representatives $M_0, \cdots, M_{d-1}$. Let L_0 be an operator called the *energy operator* which acts on the M_i and the *minimal energies* h_i be the corresponding smallest eigenvalues. For all VOAs $h_0 = 0$, and we will assume throughout that $h_i > 0$ for all $i \in \{1, \dots, d-1\}$. We define $M_i(n+h_i)$ to be the eigenspace of L_0 corresponding to the eigenvalue $n + h_i$ and the *characters* of the M_i as

$$(\mathrm{ch}M_i)(\tau) = q^{-c/24+h_i}\sum_{n\geq 0}\dim(M_i(n+h_i))\, q^n$$

where, as in the previous section, $q = e^{2\pi i\tau}$, and c is a number encoded in $\mathcal{V}$ called the *central charge*. We will assume that $c > 0$.

Define the *representation category* $\mathrm{Rep}(\mathcal{V})$ of $\mathcal{V}$ be the category of modules of $\mathcal{V}$. It was shown [Hua05] that $\mathrm{Rep}(\mathcal{V})$ is an algebraic structure called a *modular tensor category* (MTC) [EGNO15]. Encoded in $\mathrm{Rep}(\mathcal{V})$ are numbers called the *twists of simple objects* θ_i given by the relation

$$\theta_i = e^{2\pi i h_i}.$$

Thus from $\mathrm{Rep}(\mathcal{V})$ we can recover the $h_i \bmod 1$. Also encoded in $\mathrm{Rep}(\mathcal{V})$ is another number called the *multiplicative central charge* $\chi_{\mathrm{Rep}(\mathcal{V})}$ which is related to c by

$$\chi_{\mathrm{Rep}(\mathcal{V})} = e^{i\pi c/4}.$$

This allows us to recover $c \bmod 8$ from $\mathrm{Rep}(\mathcal{V})$. We define the *genus* $\mathcal{G}(\mathcal{V})$ of $\mathcal{V}$ to be the pair $(\mathrm{Rep}(\mathcal{V}), c)$. Suppose $\mathcal{C}$ is an MTC. An *admissible genus* is a pair $(\mathcal{C}, c)$ such that $\chi_{\mathcal{C}} = e^{i\pi c/4}$. If $\mathcal{C}$ is isomorphic to $\mathrm{Rep}(\mathcal{V})$, then we call $\mathcal{V}$ a *realization* of the admissible genus $(\mathcal{C}, c)$, and we call $(\mathcal{C}, c)$ *realizable* if it admits a realization.

$\mathrm{Rep}(\mathcal{V})$ induces a representation ρ of $\mathrm{SL}(2,\mathbb{Z})$ that, in our examples, is irreducible and unitary. Assuming that the modules M_i are isomorphic to their duals, ρ is then a representation of $\mathrm{PSL}(2,\mathbb{Z})$. It was proven in [Zhu96] that the *character vector* of $\mathcal{V}$

$$\mathrm{ch}(\mathcal{V}) = \begin{bmatrix} \mathrm{ch}M_0 \\ \vdots \\ \mathrm{ch}M_{d-1} \end{bmatrix}$$

is then a VVMF, i.e. an element of $\mathcal{M}(\rho)$. This means that we can use the methods of the previous section to study $\mathrm{ch}(\mathcal{V})$. If the M_i are not self-dual, we can use the trick in [BG07, Appendix A] to construct a representation ϱ of $\mathrm{PSL}(2,\mathbb{Z})$ associated to ρ. Thus we can still use the theory of VVMFs to study cases where the M_i are not self-dual.

For a VOA $\mathcal{V}$ with central charge c and minimal energies $h_0, \ldots, h_{d-1}$ it was proven in [Mas07, Sec. 3] that

$$\ell = \binom{d}{2} + \frac{dc}{4} - 6\sum_{i=0}^{d-1} h_i \in \mathbb{Z}_{\geq 0}. \tag{3}$$

We call $\mathcal{V}$ *extremal* if $\ell < 6$, and we call this condition the extremality condition. Since $\mathrm{Rep}(\mathcal{V})$ determines the $h_i \bmod 1$, this implies that the extremal $\mathcal{V}$ have maximal $\sum h_i$ of all VOAs in the genus $(\mathrm{Rep}(\mathcal{V}), c)$.

Suppose $\mathcal{V}$ is a VOA with associated representation ρ of $\mathrm{PSL}(2, \mathbb{Z})$. If the exponent matrix Λ given by

$$\Lambda_{ii} = \delta_{i0} + h_i - \frac{c}{24}, \tag{4}$$

is bijective, then $\mathbb{X}^{(1;1)}$ is the character vector of $\mathcal{V}$. Since Λ must satisfy equation (1), this further implies that there are no other character vectors of VOAs in the genus $(\mathrm{Rep}(\mathcal{V}), c)$ with $h_i' \geq h_i$ for all i.

Contrapositively, suppose $(\mathcal{C}, c)$ is an admissible genus, ρ is the representation of $\mathrm{PSL}(2, \mathbb{Z})$ associated to $\mathcal{C}$, Λ is a bijective exponent matrix satisfying $\Lambda_{00} = 1 - c/24$, and Ξ is the fundamental matrix. Then if the coefficients of the q-expansion of the first column of Ξ are not positive integers, then this column cannot be a character vector of a VOA. This implies that there is no VOA realizing $(\mathcal{C}, c)$ with minimal energies $h_i \geq \Lambda_{ii} + c/24$ for $i > 0$.

It was shown in [TW17, Thm. 3.1] the if $\mathcal{C}$ is a UMC, $(\mathcal{C}, c)$ is an admissible genus with $2 \leq \mathrm{rank}(\mathcal{C}) \leq 3$, and $\mathcal{V}$ is an extremal VOA, then the exponent matrix given by equation (4) is bijective. This implies that the character vector of $\mathcal{V}$ is uniquely determined by the minimal energies h_i. Then for an extremal choice of minimal energies h_i, if the coefficients of the q-expansion of the first column of the fundamental matrix constructed from this choice of Λ are not all positive integers, there is no extremal VOA corresponding to that choice of h_i.

6 Main Result

The goal for this section is to completely enumerate all candidate character vectors of extremal VOAs $\mathcal{V}$ with $\mathrm{rank}(\mathrm{Rep}(\mathcal{V})) = 2$. To do this, first note that for any admissible genus $(\mathcal{C}, c)$ with $\mathrm{rank}(\mathcal{C}) = 2$, because $\mathcal{C}$ fixes the $h_i \bmod 1$ and $h_0 = 0$, h_1 is completely determined by the extremality condition. This allows us to state the following definition without ambiguity.

Definition 6.1. The *extremal VOA bijective exponent* matrix of an admissible genus $(\mathcal{C}, c)$ with $\mathrm{rank}(\mathcal{C}) = 2$ is given by

$$\Lambda_{ii} = \delta_{i0} + h_i - \frac{c}{24}$$

The strategy then will be to show that there exists a recursion relation between the diagonals of the characteristic matrices χ associated to the extremal VOA bijective exponent matrix for the genera $(\mathcal{C}, c)$ and $(\mathcal{C}, c + 24)$.

We will then show that this recursion relation implies that, as c increases, eventually χ_{11} will no longer be a positive integer. Since $c \bmod 8$ is fixed by $\mathcal{C}$, once the method explained in [BG07] is used to calculate the characteristic matrices of 3 *seed genera*, only finitely many candidates remain for $\mathcal{C}$. Since it was shown in [RSW09] that there are only 4 MTCs $\mathcal{C}$ with $\operatorname{rank}(\mathcal{C}) = 2$, we have thus reduced the problem to a finite set of computations which can be done on a computer.

Lemma 6.2. *Let $(\mathcal{C}, c)$ be an admissible genus with $\operatorname{rank}(\mathcal{C}) = 2$. Let Λ_- and Λ_+ be the extremal VOA bijective exponent matrices for the genera $(\mathcal{C}, c)$ and $(\mathcal{C}, c+24)$ respectively. Then*

$$[\Lambda_+]_{ii} = [\Lambda_-]_{ii} + (-1)^{i+1}.$$

Proof. Since $h_0 = 0$, we have

$$[\Lambda_+]_{00} = 1 - \frac{c+24}{24} = [\Lambda_-]_{00} - 1.$$

Since the representation ρ of $\mathrm{PSL}(2, \mathbb{Z})$ is the same for both genera, equation (1) fixes $\operatorname{Tr}(\Lambda_+) = \operatorname{Tr}(\Lambda_-)$, so

$$(\Lambda_+)_{11} = \operatorname{Tr}(\Lambda_+) - (\Lambda_+)_{00} = \operatorname{Tr}(\Lambda_-) - (\Lambda_-)_{00} + 1 = (\Lambda_-)_{11} + 1.$$

□

Corollary 6.3. *The non-trivial minimal energies of $(\mathcal{C}, c)$ and $(\mathcal{C}, c+24)$, denoted h_- and h_+, respectively, are related by $h_+ = h_- + 2$.*

Lemma 6.4. *Let everything be defined as in Lemma 6.2 and Corollary 6.3. Let χ_- and χ_+ be the characteristic matrices constructed from Λ_- and Λ_+, respectively, and ρ be the representation of $PSL(2, \mathbb{Z})$ associated with $\mathcal{C}$. Then*

$$[\chi_+]_{11} = \frac{[\chi_-]_{22} + h_-([\chi_-]_{11} - 240)}{h_- + 1}$$
$$[\chi_+]_{22} = \frac{[\chi_-]_{11} + h_-([\chi_-]_{22} + 240)}{h_- + 1}.$$

Note that these statements are well-defined because we have been assuming that $h_- > 0$.

Proof. Let $\Xi_-(q)$ and $\Xi_+(q)$ be the q-expansions of the fundamental matrices corresponding to Λ_- and Λ_+ respectively. Then

$$\begin{aligned} q^{-\Lambda_-}\Xi_-(q) &= q^{-\Lambda_-}[\mathbb{X}_-^{(1;1)}(q)|\mathbb{X}_-^{(2;1)}(q)] \\ &= \begin{bmatrix} q^{-1}+a_{11}^{(1)}+a_{11}^{(2)}q+\dots & a_{12}^{(1)}+a_{12}^{(2)}q+\cdots \\ a_{21}^{(1)}+a_{21}^{(2)}q+\dots & q^{-1}+a_{22}^{(1)}+a_{22}^{(2)}q+\cdots \end{bmatrix} \end{aligned}$$

and similarly

$$\begin{aligned} q^{-\Lambda_+}\Xi_+(q) &= q^{-\Lambda_+}[\mathbb{X}_+^{(1;1)}(q)|\mathbb{X}_+^{(2;1)}(q)] \\ &= \begin{bmatrix} q^{-1}+b_{11}^{(1)}+b_{11}^{(2)}q+\dots & b_{12}^{(1)}+b_{12}^{(2)}q+\cdots \\ b_{21}^{(1)}+b_{21}^{(2)}q+\dots & q^{-1}+b_{22}^{(1)}+b_{22}^{(2)}q+\cdots \end{bmatrix} \end{aligned}$$

where $a_{ij}^{(k)}, b_{ij}^{(k)} \in \mathbb{C}$ for all $i, j \in \{1, 2\}$ and $k \in \mathbb{N}$.

Then, by Lemma 6.2, we have

$$\begin{aligned} q^{-\Lambda_-}\Xi_+(q) &= q^{\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} q^{-\Lambda_+}\Xi_+(q) \\ &= \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} q^{-1}+b_{11}^{(1)}+b_{11}^{(2)}q+\dots & b_{12}^{(1)}+b_{12}^{(2)}q+\cdots \\ b_{21}^{(1)}+b_{21}^{(2)}q+\dots & q^{-1}+b_{22}^{(1)}+b_{22}^{(2)}q+\cdots \end{bmatrix} \\ &= \begin{bmatrix} q^{-2}+b_{11}^{(1)}+b_{11}^{(2)}q^{-1}+\dots & b_{12}^{(1)}q^{-1}+b_{12}^{(2)}+\cdots \\ b_{21}^{(1)}q+b_{21}^{(2)}q^2+\dots & 1+b_{22}^{(1)}+b_{22}^{(2)}q+\cdots \end{bmatrix}. \end{aligned}$$

Since the sets $\{\mathbb{X}_-^{(1;1)}, \mathbb{X}_-^{(2,1)}\}$ and $\{\mathbb{X}_+^{(1;1)}, \mathbb{X}_+^{(2,1)}\}$ are both bases for $\mathcal{M}(\rho)$ as a free $\mathbb{C}[J]$-module, this implies that $a_{21}^{(1)}\mathbb{X}_+^{(2;1)} = \mathbb{X}_-^{(1;1)}$. Since $\mathbb{X}_-^{(1;1)} \neq 0$, we see that $a_{21}^{(1)} \neq 0$, so

$$\mathbb{X}_+^{(2;1)} = \frac{1}{a_{21}^{(1)}}\mathbb{X}_-^{(1;1)}.$$

In addition, the fact that $\mathcal{M}(\rho)$ is a free $\mathbb{C}[J]$-module implies that

$$\mathbb{X}_+^{(1;1)} = A(J)\mathbb{X}_-^{(1;1)} + B(J)\mathbb{X}_-^{(2;1)}$$

for some $A(J), B(J) \in \mathbb{C}[J]$. Left multiplying both sides by $q^{-\Lambda_-}$ yields

$$\begin{aligned} \begin{bmatrix} q^{-2}+b_{11}^{(1)}q^{-1}+\cdots \\ b_{21}^{(1)}q+b_{21}^{(2)}q^2+\cdots \end{bmatrix} =& A(J)\begin{bmatrix} q^{-1}+a_{11}^{(1)}+\cdots \\ a_{21}^{(2)}+a_{21}^{(2)}q+\cdots \end{bmatrix} \\ &+ B(J)\begin{bmatrix} a_{12}^{(1)}+a_{12}^{(2)}q+\cdots \\ q^{-1}+a_{22}^{(1)}+\cdots \end{bmatrix}. \end{aligned}$$

Since $J(q)$ has a simple pole, we quickly find

$$A(J) = J + a_{22}^{(1)} - \frac{a_{21}^{(2)}}{a_{21}^{(1)}}, \quad B(J) = -a_{21}^{(1)}.$$

All together this gives

$$\Xi_+ = \left[\left(J + a_{22}^{(1)} - \frac{a_{21}^{(2)}}{a_{21}^{(1)}} \right) \mathbb{X}_-^{(1;1)} - a_{21}^{(1)} \mathbb{X}_-^{(2;1)} \middle| \frac{1}{a_{21}^{(1)}} \mathbb{X}_-^{(1;1)} \right]$$

Now we calculate

$$\begin{aligned}
\left[q^{-\Lambda_+} \mathbb{X}_+^{(1;1)}(q) \right]_1 &= \left[q^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]} q^{-\Lambda_-} \mathbb{X}_+^{(1;1)}(q) \right]_1 \\
&= q \left(J(q) + a_{22}^{(1)} - \frac{a_{21}^{(2)}}{a_{21}^{(1)}} \right) \left[q^{-\Lambda_-} \mathbb{X}_-^{(1;1)}(q) \right]_1 \\
&\quad - a_{21}^{(1)} q \left[q^{-\Lambda_-} \mathbb{X}_-^{(2;1)}(q) \right]_1 \\
&= q \left(q^{-2} + \left(a_{11}^{(1)} + a_{22}^{(1)} - \frac{a_{21}^{(2)}}{a_{21}^{(1)}} \right) q^{-1} + O(1) \right) \\
&\quad - a_{21}^{(1)} q \left(a_{12}^{(1)} + a_{12}^{(2)} q + O(q^2) \right) \\
&= q^{-1} + \left(a_{11}^{(1)} + a_{22}^{(1)} - \frac{a_{21}^{(2)}}{a_{21}^{(1)}} \right) + O(q),
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\left[q^{-\Lambda_+} \mathbb{X}_+^{(2;1)}(q) \right]_2 &= \left[q^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]} q^{-\Lambda_-} \mathbb{X}_+^{(2;1)}(q) \right]_2 \\
&= \frac{1}{a_{21}^{(1)}} q^{-1} \left[q^{-\Lambda_-} \mathbb{X}_-^{(1;1)} \right]_2 \\
&= \frac{1}{a_{21}^{(1)}} q^{-1} \left(a_{21}^{(1)} + a_{21}^{(2)} q + O(q^2) \right) \\
&= \frac{a_{21}^{(1)}}{a_{21}^{(1)}} q^{-1} + \frac{a_{21}^{(2)}}{a_{21}^{(1)}} + O(q).
\end{aligned}$$

By the chain rule

$$\frac{1}{2\pi i}\frac{d\Xi(\tau)}{d\tau} = q\frac{d\Xi(q)}{dq},$$

so, using the recursion relation given in equation (2), we find that

$$\begin{aligned} 0 &= \left[q^{-\Lambda_-}\Big(q\frac{d\Xi_+(q)}{dq} - \Xi_+(q)\mathcal{D}(q)\Big)\right]_{21} \\ &= \Big(- a_{21}^{(1)}\Big(a_{11}^{(1)} + h_-(a_{22}^{(1)} + 240)\Big) + a_{21}^{(2)}(h_- + 1)\Big)q + \ldots. \end{aligned}$$

Rearranging this first term gives us

$$\frac{a_{21}^{(2)}}{a_{21}^{(1)}} = \frac{a_{11}^{(1)} + h_-(a_{22}^{(1)} + 240)}{h_- + 1}$$

Thus we have that

$$[\chi_+]_{11} = a_{11}^{(1)} + a_{22}^{(1)} - \frac{a_{21}^{(2)}}{a_{21}^{(1)}} = \frac{a_{22}^{(1)} + h_-(a_{11}^{(1)} - 240)}{h_- + 1},$$

and

$$[\chi_+]_{22} = \frac{a_{21}^{(2)}}{a_{21}^{(1)}} = \frac{a_{11}^{(1)} + h_-(a_{22}^{(1)} + 240)}{h_- + 1}$$

Since $[\chi_-]_{11} = a_{11}^{(1)}$ and $[\chi_-]_{22} = a_{22}^{(1)}$, this completes the proof. □

Thus the diagonals of χ_+ can be completely solved for in terms of only the diagonals of χ_-. This means that we can recursively solve for the diagonals of the characteristic matrix corresponding to any central charge $c+24n$ with $n \in \mathbb{N}$. We can now think of the map f that takes the diagonals of χ_- to the diagonals of χ_+ as a discrete dynamical system and investigate its long term behavior.

Lemma 6.5. *Let*

$$f(x,y,h) = \Big(\tfrac{y+h(x-240)}{h+1}, \quad \tfrac{x+h(y+240)}{h+1}, \quad h+2\Big),$$

where $h \in \mathbb{R}^+$ and $x, y \in \mathbb{C}$. Then for all $n \in \mathbb{N}$

$$f^n(x,y,h) = \Big(\tfrac{ny+(h+n-1)(x-240n)}{h+2n-1}, \quad \tfrac{nx+(h+n-1)(y+240n)}{h+2n-1}, \quad h+2n\Big), \qquad (5)$$

where the superscript here denotes iteration, not differentiation.

Proof. The result clearly holds for $n = 1$. We proceed by induction on n. Suppose by way of induction that equation (5) holds for some value of $n \in \mathbb{N}$. Then

$$\begin{aligned}
[f^{n+1}(x,y,h)]_1 =& [f(f^n(x,y,h))]_1 \\
=& \left[f\left(\tfrac{ny+(h+n-1)(x-240n)}{h+2n-1}, \quad \tfrac{nx+(h+n-1)(y+240n)}{h+2n-1}, \quad h+2n \right) \right]_1 \\
=& \frac{1}{h+2n+1} \left[\frac{nx+(h+n-1)(y+240n)}{h+2n-1} \right. \\
& \left. + (h+2n)\left(\frac{ny+(h+n-1)(x-240n)}{h+2n-1} - 240 \right) \right] \\
=& \frac{1}{(h+2n+1)(h+2n-1)} \left[(h^2+3hn+2n^2-h-n)x \right. \\
& + (hn+2n^2+h+n-1)y \\
& \left. - 240(h^2n+3hn^2+2n^3+h^2+2hn+n^2+h+n) \right] \\
=& \frac{(h+n)x+(n+1)y-240(h+n)(n+1)}{h+2n+1} \\
=& \frac{(n+1)y+[h+(n+1)-1][x-240(n+1)]}{h+2(n+1)-1}.
\end{aligned}$$

Switching the roles of x and y and replacing 240 with -240 in the above calculation shows that the hypothesis holds for $[f^{n+1}(h,x,y)]_2$. It is obvious that $[f^{n+1}(h,x,y)]_3 = h+2(n+1)$, so equation (5) has been proven for all $n \in \mathbb{N}$. □

Now we simply put all of these lemmas together to get the main theorem.

Theorem 6.6. *Suppose $(\mathcal{C}, c)$ is an admissible genus with rank$(\mathcal{C}) = 2$ and non-trivial extremal minimal energy h. For $n \in \mathbb{N}$, let χ_n denote the character matrix constructed from the extremal VOA bijective exponent associated with the genus $(\mathcal{C}, c+24n)$. Then for*

$$\begin{aligned}
n >& \frac{1}{480}\Bigg([\chi_0]_{11} + [\chi_0]_{22} - 240(h-1) \\
& + \sqrt{\Big([\chi_0]_{11} + [\chi_0]_{22} - 240(h-1)\Big)^2 + 960(h-1)[\chi_0]_{11}} \Bigg), \qquad (6)
\end{aligned}$$

$[\chi_n]_{11}$ is not a positive integer, so $(\mathcal{C}, c+24n)$ is not realizable.

Proof. First, it is easy to directly compute the characteristic matrices for all possible seed genera, showing us that each $[\chi_0]_{ij} \in \mathbb{Q}$. We define f as in Lemma 6.5. Then, by Lemma 6.4, we have

$$\begin{aligned}[\chi_n]_{11} =& \Big[f^n\big([\chi_0]_{11}, [\chi_0]_{22}, h\big)\Big]_1 \\ =& \frac{n[\chi_0]_{22} + (h+n-1)([\chi_0]_{11} - 240n)}{h+2n-1} \\ =& \frac{-240n^2 + \Big([\chi_0]_{11} + [\chi_0]_{22} - 240(h-1)\Big)n + (h-1)[\chi_0]_{11}}{2n+h-1},\end{aligned}$$

where the last line has just been rewritten in a more suggestive form. We consider the numerator and denominator of $[\chi_n]_{11}$ as functions of n. Since $h > 0$, the denominator is greater than 0 for all $n \in \mathbb{N}$. The condition on n given in equation (6) is guarantees that n is larger than both roots of the numerator. Since the numerator is concave down, this implies that for n satisfying (6), $[\chi_n]_{11} < 0$. Thus $[\chi_n]_{11}$ is not a positive integer. □

This theorem gives an upper bound on the number of possible admissible genera constructed in this way from the seed genus $(\mathcal{C}, c)$. The fundamental matrices constructed from the extremal VOA bijective matrices of these finite admissible genera can then be computed individually.

Example 6.7. Let $\mathcal{C} = \mathrm{Rep}(\mathrm{SU}(2)_1)$. For this choice of $\mathcal{C}$, $\theta_1 = i$ and $\chi_{\mathcal{C}} = (1+i)/\sqrt{2}$. Thus $h_1 \bmod 1 = 1/4$ and $c \bmod 8 = 1$, so our seed genera will be $c = 1, 9, 17$. Fixing $c = 9$, we see that

$$\ell = \binom{2}{2} + \frac{2 \cdot 9}{4} - 6\sum_{i=0}^{1} h_i = \frac{11}{2} - 6h_1.$$

The extremality condition, $0 \leq \ell < 6$, then shows that $-1/2 < h_1 \leq 11/12$, so $h_1 = 1/4$. The extremal VOA bijective exponent matrix is then given by

$$\Lambda = \begin{bmatrix} 1 - \frac{9}{24} & 0 \\ 0 & \frac{1}{4} - \frac{9}{24} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & 0 \\ 0 & \frac{-1}{8} \end{bmatrix}.$$

$\mathcal{C}$ also fixes the representation ρ of $\mathrm{PSL}(2\mathbb{Z})$ by

$$\rho(S) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Let χ_n denote the characteristic matrix constructed from ρ and the extremal VOA bijective exponent associated with the genus $(\mathrm{Rep}(\mathrm{SU}(2)_1), 9+24n)$. Then the method laid out in [BG07] allows us to calculate

$$\chi_0 = \begin{bmatrix} 251 & 26752 \\ 2 & 1 \end{bmatrix}$$

Then equation (6) becomes

$$\begin{aligned} n >& \frac{1}{480}\Bigg((251+1) - 240\Big(\frac{1}{4}-1\Big) \\ &+ \sqrt{\Big(251+1-240\Big(\frac{1}{4}-1\Big)\Big)^2 + 960\Big(\frac{1}{4}-1\Big)251}\Bigg) \\ =& \frac{432+12\sqrt{41}}{480}. \end{aligned}$$

This means that $(\mathrm{Rep}(\mathrm{SU}(2)_1), 9+24n)$ is not realizable for

$$n > \left\lfloor \frac{432+12\sqrt{41}}{480} \right\rfloor = 1.$$

The only case to check then is $n=1$, i.e. $c=33$. We can again directly calculate

$$\chi_1 = \begin{bmatrix} 3 & \frac{1}{2} \\ 565760 & 249 \end{bmatrix},$$

which does in fact have all positive integers in the first column. As a sanity check we calculate

$$\chi_2 = \frac{1}{13}\begin{bmatrix} -1137 & \frac{1}{43520} \\ 1412731371520 & 4413 \end{bmatrix}.$$

Thus $(\mathrm{Rep}(\mathrm{SU}(2)_1), 57)$ is not realizable, as we expected. The cases $c = 1, 17$ proceed in the same way.

As mentioned at the beginning of this section, the choice of $\mathcal{C}$ fixes $c \bmod 8$, so there are only 3 unique seed genera for each $\mathcal{C}$. In addition, there are only 4 MTCs $\mathcal{C}$ of $\mathrm{rank}(\mathcal{C}) = 2$. Thus we have shown that the number of admissible genera that are possibly realizable to be finite. Having repeated the above calculations for all modular tensor categories of rank 2, we have found no candidate character vectors that were not listed in [TW17, Sec. 3.2]. In section 7 we recreate their list of candidate character vectors and realizations (where a realization is known).

For future study, we hope to construct the realizations for all the candidates listed Section 7. In addition, we hope to extend our results to extremal VOAs $\mathcal{V}$ with rank(Rep($\mathcal{V}$)) = 3, as the results of [TW17] still apply to this class of VOAs, allowing us to use the theory of VVMFs to study them as well.

7 Data

Below is a reproduction of the complete list of the candidate extremal character vectors for MTCs $\mathcal{C}$ of rank($\mathcal{C}$) = 2 given in [TW17]. We include the extremal VOAs that realize the character vectors where they are known. The one case in which there is no known extremal VOA realizing the candidate is marked "unknown".

7.1 $\mathcal{C} = \mathbf{Rep}(\mathbf{SU}(2)_1)$

$$\theta_1 = i, \qquad \chi_{\mathcal{C}} = \frac{1+i}{\sqrt{2}}, \qquad \rho(S) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

c	h_1	χ	Character Vector	Realization
1	1/4	$\begin{bmatrix} 3 & 26752 \\ 2 & -247 \end{bmatrix}$	$q^{-1/24}\begin{bmatrix} 1+3q+4q^2+\cdots \\ q^{1/4}(2+2q+6q^2+\cdots) \end{bmatrix}$	$\mathrm{SU}(2)_1$
9	1/4	$\begin{bmatrix} 251 & 26752 \\ 2 & 1 \end{bmatrix}$	$q^{-3/8}\begin{bmatrix} 1+251q+4872q^2+\cdots \\ q^{1/4}(2+498q+8750q^2+\cdots) \end{bmatrix}$	$\mathrm{SU}(2)_1 \otimes E_{8,1}$
17	5/4	$\begin{bmatrix} 323 & 88 \\ 1632 & -319 \end{bmatrix}$	$q^{-17/24}\begin{bmatrix} 1+323+60860q^2+\cdots \\ q^{5/4}(1632+162656q+\cdots \end{bmatrix}$	[GHM16]
33	9/4	$\begin{bmatrix} 3 & \frac{1}{2} \\ 565760 & 249 \end{bmatrix}$	$q^{-11/8}\begin{bmatrix} 1+3q+86004q^2+\cdots \\ q^{9/4}(565760+192053760q+\cdots) \end{bmatrix}$	UNKNOWN

7.2 $\mathcal{C} = \mathbf{Rep}(E_{7,1})$

$$\theta_1 = -i, \qquad \chi_{\mathcal{C}} = \frac{1-i}{\sqrt{2}}, \qquad \rho(S) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

c	h_1	χ	Character Vector	Realization
7	$3/4$	$\begin{bmatrix} 133 & 1248 \\ 56 & -377 \end{bmatrix}$	$q^{-7/24}\begin{bmatrix} 1+133q+1673q^2+\cdots \\ q^{3/4}(56+968q+7506q^2+\cdots) \end{bmatrix}$	$E_{7,1}$
15	$3/4$	$\begin{bmatrix} 381 & 1248 \\ 56 & -129 \end{bmatrix}$	$q^{-5/8}\begin{bmatrix} 1+381q+38781q2+\cdots \\ q^{3/4}(56+14856q+478512q^2+\cdots) \end{bmatrix}$	$E_{7,1}\otimes E_{8,1}$
23	$7/4$	$\begin{bmatrix} 69 & 10 \\ 32384 & -65 \end{bmatrix}$	$q^{-23/24}\begin{bmatrix} 1+69q+131905q^2+\cdots \\ q^{7/4}(32384+4418944q+\cdots) \end{bmatrix}$	[GHM16]

7.3 $\mathcal{C} = \mathbf{Rep}(G_{2,1})$

$$\theta_1 = e^{4\pi i/5}, \qquad \chi_{\mathcal{C}} = e^{-i\pi/20}, \qquad \rho(S) = \frac{1}{\sqrt{2+\phi}}\begin{bmatrix} 1 & \phi \\ \phi & -1 \end{bmatrix}, \qquad \phi = \frac{1+\sqrt{5}}{2}$$

c	h_1	χ	Character Vector	Realization
$14/5$	$2/5$	$\begin{bmatrix} 14 & 12857 \\ 7 & -259 \end{bmatrix}$	$q^{-7/60}\begin{bmatrix} 1+14q+42q^2+\cdots \\ q^{2/5}(7+34q+119q^2+\cdots) \end{bmatrix}$	$G_{2,1}$
$54/5$	$2/5$	$\begin{bmatrix} 262 & 12857 \\ 7 & -10 \end{bmatrix}$	$q^{-9/20}\begin{bmatrix} 1+262q+7638q^2+\cdots \\ q^{2/5}(7+1770q+37419q^2+\cdots) \end{bmatrix}$	$G_{2,1}\otimes E_{8,1}$
$94/5$	$7/5$	$\begin{bmatrix} 188 & 46 \\ 4794 & -184 \end{bmatrix}$	$q^{-47/60}\begin{bmatrix} 1+188q+62087q^2+\cdots \\ q^{7/5}(4794+532134q+\cdots) \end{bmatrix}$	[GHM16]

7.4 $\mathcal{C} = \mathbf{Rep}(F_{4,1})$

$$\theta_1 = e^{6\pi i/5}, \qquad \chi_{\mathcal{C}} = e^{i\pi/20}, \qquad \rho(S) = \frac{1}{\sqrt{2+\phi}}\begin{bmatrix} 1 & \phi \\ \phi & -1 \end{bmatrix}, \qquad \phi = \frac{1+\sqrt{5}}{2}$$

c	h_1	χ	Character Vector	Realization
$26/5$	$3/5$	$\begin{bmatrix} 52 & 3774 \\ 26 & -296 \end{bmatrix}$	$q^{-13/60}\begin{bmatrix} 1+52q+377q^2+\cdots \\ q^{3/5}(26+299q+1702q^2+\cdots) \end{bmatrix}$	$F_{4,1}$
$66/5$	$3/5$	$\begin{bmatrix} 300 & 26 \\ 3774 & -48 \end{bmatrix}$	$q^{-11/20}\begin{bmatrix} 1+300q+17397q^2+\cdots \\ q^{3/5}(26+6747q+183078q^2+\cdots) \end{bmatrix}$	$F_{4,1}\otimes E_{8,1}$
$106/5$	$8/5$	$\begin{bmatrix} 106 & 15847 \\ 17 & -102 \end{bmatrix}$	$q^{-53/60}\begin{bmatrix} 1+106q+84429q^2+\cdots \\ q^{8/5}(15847+1991846q+\cdots) \end{bmatrix}$	[GHM16]

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