Invariants in the II_1 subfactor theory of von Neumann algebras

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For my parents.

Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

A II_1 subfactor is a pair of von Neumann algebras $N \subset M$ with trivial centre and equipped with a *trace*. It is associated to an *index* $[M : N]$, an invariant that can take non-integer values. We prove the *Jones index theorem*, a major result of Vaughan Jones stating that the set of all possible indices is $\{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty]$. Our proof incorporates the theory of *Temperley-Lieb algebras*.

We also construct the *principal graph* of a II_1 subfactor. We prove that it strictly generalises the index, and obtain a second proof of the index theorem. We compute the index and principal graph for an important family of subfactors called the *Jones subfactors*.

Contents

Notation and terminology

Notation

$\mathcal{B}(\mathcal{H})$	The set of bounded operators on a Hilbert space $\mathcal{H}$.
S'	The commutant of S on some space $\mathcal{H}$, i.e. $S' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx \text{ for all } y \in S\}$.
$\mathrm{SOT}, \mathrm{WOT}$	Strong operator topology; weak operator topology.
$\text{s-lim}, \text{w-lim}$	Strong limit; weak limit.
$\langle S \rangle_{\mathrm{alg}}$, $\langle S \rangle$	The $*$-algebra generated by a set S; the von Neumann algebra generated by a set S.
$p \vee q, p \wedge q$	The projection onto $\operatorname{im} p + \operatorname{im} q$; the projection onto $\operatorname{im} p \cap \operatorname{im} q$.
$\bigvee_{p \in S} p, \bigwedge_{p \in S} p$	The projection onto $\overline{\sum_{p \in S} \operatorname{im} p}$; the projection onto $\bigcap_{p \in S} \operatorname{im} p$.
$p \leq q$	$\operatorname{im} p \subset \operatorname{im} q$.
$\preceq, \approx$	Murray-von Neumann order; Murray-von Neumann equivalence (Definition 1.3.2).
$\vec{n}^A, \Lambda_A^B, \beta_A^B$	Dimension vector of A; inclusion matrix of $A \subset B$; Bratteli diagram of $A \subset B$. (Definitions 1.5.6, 1.5.8, 1.5.10)
$V(G)$	Vertex set of a graph G.
$\hat{M}, \hat{x}$	M viewed as a vector space; an element $x \in M$ viewed as an element of $\hat{M}$.
$L^2(M)$	The standard form Hilbert space for M, containing $\hat{M}$ as a dense subspace (Theorem 1.6.5).

J	$J : L^2(M) \to L^2(M)$ is the extension of the adjoint map $\hat{x} \mapsto \widehat{(x^*)}$.
$\|\cdot\|$	The operator norm on a von Neumann algebra. Not to be confused with the L^2-norm.
$\langle \cdot, \cdot \rangle_{L^2}$, $\|\cdot\|_{L^2}$	$\langle \hat{x}, \hat{y} \rangle_{L^2} = \mathrm{tr}(y^* x)$; $\|\hat{x}\|_{L^2} = \mathrm{tr}(x^* x)^{1/2}$. Not to be confused with the operator norm.
e_N	The projection of $L^2(M)$ onto $L^2(N)$, called the Jones projection.
E_N	The unique trace-preserving conditional expectation of M onto N.
$\langle M, e_N \rangle$ or M_1	The basic construction of $N \subset M$.
M_n	$M_{-1} = N, M_0 = M$, and M_n is the basic construction of $M_{n-2} \subset M_{n-1}$.
e_n	Notation for $e_{M_{n-2}} : L^2(M_{n-1}) \to L^2(M_{n-2})$. Note e_n belongs to M_n.
ε_n	A member of a family of projections satisfying the Jones relations.
$[n : m]$	Notation for $\langle 1, \varepsilon_n, \varepsilon_{n+1}, \ldots, \varepsilon_m \rangle_{\mathrm{alg}}$.
E_n	A generator of the Temperley-Lieb algebra.
$TL(\tau), TL_n(\tau)$	The Temperley-Lieb algebra; the $*$-subalgebra of $TL(\tau)$ generated by $1, E_1, \ldots, E_n$.
$J^{(n)} \subset J$	The Jones subfactor of index $4\cos^2(\pi/n)$ (Definition 2.8.5).
Y_n	Notation for $N' \cap M_n$.
z_n	Notation for $z_{Y_n}(e_n)$.
X_n	Notation for $Y_n z_n$.
β_n^{n+1}, Λ_n^{n+1}	The Bratteli diagram of $Y_n \subset Y_{n+1}$; the inclusion matrix.
β, Γ	The full Bratteli diagram of a subfactor (Definition 3.4.1); the principal graph of a subfactor (Definition 3.5.1).

P_n, P_n^{new}, $\tilde{P}_n$	The vertices of Γ at level n; the new vertices at level n; the old vertices at level n.

Terminology

$*$-algebra	A $\mathbb{C}$-vector space equipped with a $\mathbb{C}$-linear multiplication and antilinear involution $* : A \to A$ denoted by $a \mapsto a^*$.
$*$-homomorphism	A linear map of $*$-algebras $\psi : A \to B$ that respects the multiplication and the $*$-operation.
Unital	A unital $*$-subalgebra $A \subset B$ is such that A contains the identity of B. A unital map is a map that preserves the identity.

Introduction

The theory of II_1 subfactors is a subfield of *von Neumann algebra theory*. Introduced by Francis Murray and John von Neumann [MV36], von Neumann algebras are $*$-algebras of operators on a Hilbert space which contain 1 and are closed in the weak operator topology. They are significant in analysis as an important setting for the spectral theory of bounded operators; they are also important to formalisations of quantum mechanics.

Among the von Neumann algebras studied by Murray and von Neumann are the II_1 *factors*. They are the family of infinite-dimensional von Neumann algebras which are both *factors* (having trivial centre) and *tracial* (being equipped with a nice trace). They generalise the structure of $L^\infty(X, \mu)$ and its trace $f \mapsto \int f d\mu$. The interactions between traces and algebraic structure confers II_1 factors with far more structure than the sum of their parts.

A II_1 subfactor is an inclusion of II_1 factors $N \subset M$. In fact, all morphisms of II_1 factors are injective, so II_1 subfactor theory is equivalent to the theory of morphisms of II_1 factors. Unlike II_1 factor theory, II_1 *subfactor theory* came to maturity well after Murray and von Neumann. Vaughan Jones's groundbreaking paper 'Index for Subfactors' [Jon83] signifies the beginning of modern II_1 subfactor theory, and led to an explosion of new constructions and invariants. His work [Jon83] [Jon85] [Jon87] also revealed connections between II_1 factors and apparently unrelated fields, including statistical mechanics, quantum algebras, and knot theory.

In this thesis, we introduce two of the first modern invariants of a II_1 subfactor, and prove a major theorem using each. We introduce the index $[M : N]$ and prove the *Jones index theorem*. This theorem states that, unlike the index of a subgroup, the index $[M : N]$ takes a value in $\{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty]$. We give a modernised proof with techniques from *Temperley-Lieb algebra theory*, a subfield of quantum algebra.

We also construct an invariant which appeared later in the literature than the index [Ocn88] [GHJ89] [Pop90] – the *principal graph* Γ. We prove that the principal graph generalises the index, allowing us to give a second, *graph-theoretic proof* of the Jones index theorem. We also compute the principal graphs for an important family of subfactors known as *Jones subfactors*.

Structure of the thesis

In Chapter 1, we present a rapid overview of the background theory necessary for II_1 subfactor theory. We largely omit proofs, as almost all of this material is 'classical', originating in Murray and von Neumann's papers [MV36] [MN37] [Neu40] [MN43] or coming not long after. The exception is the *index* $[M : N]$ of a II_1 subfactor $N \subset M$, which was introduced by Jones [Jon83].

In Chapter 2, we introduce and prove the Jones index theorem. Towards the proof, we describe the most important technical innovations of Jones's paper [Jon83]: the *basic construction*, a technique that extends a subfactor $N \subset M$ to a triplet $N \subset M \subset M_1$, as well as the *Jones tower*, the sequence $\{M_n\}$ obtained by iterating the construction indefinitely.

We show that a *Temperley-Lieb algebra* arises in the Jones tower. We develop tools for probing Temperley-Lieb algebras, and use them to prove the index theorem.

In Chapter 3, we give an account of the *principal graph*, an invariant that generalises the index and has become central to II_1 subfactor classification efforts. We first introduce the *standard invariant*, a major invariant for II_1 subfactors that is built from the Jones tower. From the standard invariant, we construct the principal graph Γ. We prove, in certain circumstances, Γ generalises the index via the relation $\|\Gamma\|^2 = [M : N]$ and hence obtain an alternate proof of the Jones index theorem.

Finally, we provide computations of the three major invariants (standard invariant, principal graph, index) for the *Jones subfactors*, the first family of subfactors to be constructed with modern techniques.

Overall, this thesis acts as an account of the explosion of modern subfactor theory in the decade following 'Index for Subfactors' [Jon83].

Chapter 1

Background

1.1 Introduction

In this chapter, we cover the background necessary for a study of modern II_1 subfactor theory. We cover general theory of von Neumann algebras, building up to the definition and basic properties of II_1 factors. Much of this material originates in the work of Murray and von Neumann. We largely sketch or omit proofs.

We introduce one concept from II_1 subfactor theory in this chapter – the *index* $[M : N]$, a relative measure of size for a subfactor $N \subset M$, introduced by Vaughan Jones [Jon83].

1.2 Definition and elementary notions

In this chapter, $\mathcal{H}$ and $\mathcal{K}$ denote arbitrary Hilbert spaces. All Hilbert spaces in this thesis are assumed to be separable.

Definition 1.2.1. Let $S \subset \mathcal{B}(\mathcal{H})$. The *commutant* of S, denoted by S', is defined by $S' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx \text{ for all } y \in S\}$. The *bicommutant* of S, denoted by S'', is the commutant of S'.

Remark 1.2.2. If S, T are any subsets of $\mathcal{B}(\mathcal{H})$, then $S \subset T \implies T' \subset S'$. Moreover, $S \subset S''$.

We recall the definitions of four important operator topologies. In the weak operator topology on $\mathcal{B}(\mathcal{H})$, a net $\{x_\alpha\} \subset \mathcal{B}(\mathcal{H})$ converges to $x \in \mathcal{B}(\mathcal{H})$ if and only if $\langle x\xi, \eta\rangle \xrightarrow[\alpha\to\infty]{} \langle x\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. In the strong operator topology,

x_α converges to x if and only if $x_\alpha\xi \xrightarrow[\alpha\to\infty]{} x\xi$ for all $\xi \in \mathcal{H}$. The ultraweak (ultrastrong) topology is the weak (strong) topology on $\mathcal{B}(\mathcal{H}) \otimes 1 \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$ for $\mathcal{L}$ a Hilbert space with $\dim \mathcal{L} = \infty$, carried over to $\mathcal{B}(\mathcal{H})$ by the obvious isomorphism $x \otimes 1 \mapsto x$ [Jon09, p18].

Lemma 1.2.3. *Suppose A is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the closures of M in any of the (ultra)weak and (ultra)strong topologies coincide.*

See [Jon09, p20]. Hence we write $\overline{A}$ to unambiguously denote the closure of A in any of the four topologies. If A is a $*$-subalgebra containing the identity, a stronger result applies: the famous *von Neumann bicommutant theorem.*

Theorem 1.2.4. *(von Neumann bicommutant theorem) [MV36]*

Suppose M is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1. Then the closures of M in any of the (ultra)weak and (ultra)strong topologies coincide with M''.

See [Bla06, p49] for a proof. This result is remarkable, because the closure is analytic, while M'' is defined algebraically. We hence have equivalent analytic and algebraic definitions of a von Neumann algebra.

Definition 1.2.5. A *von Neumann algebra M* on $\mathcal{H}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1 and satisfying, equivalently,

1. $M = M''$.

2. M is closed in one (and hence all) of the (ultra)weak and (ultra)strong topologies.

As the (ultra)weak and (ultra)strong topologies are all weaker than the operator norm topology, a von Neumann algebra is closed in the norm topology and therefore a C^*-algebra. (No C^*-algebra theory is needed for this thesis.)

Definition 1.2.6. The von Neumann algebra generated by a set $S \subset \mathcal{B}(\mathcal{H})$, denoted by $\langle S \rangle$, is equivalently defined as

1. $\langle S \rangle = (S \cup S^*)''$.

2. $\langle S \rangle = \overline{\langle S \cup \{1\} \rangle_{\text{alg}}}$, where $\langle S \cup \{1\} \rangle_{\text{alg}}$ is the $*$-algebra generated by $S \cup \{1\}$ and the closure can be taken in the (ultra)strong or (ultra)weak topologies.

Example 1.2.7. $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra for any Hilbert space $\mathcal{H}$.

Example 1.2.8. If (X, μ) is a finite measure space, the $*$-algebra of multiplication operators $M_f : g \mapsto fg$ for $f \in L^\infty(X, \mu)$ is a von Neumann algebra, which we identify with $L^\infty(X, \mu)$ itself.

In fact, all abelian von Neumann algebras are isomorphic to $L^\infty(X, \mu)$ for some X. Depending on one's perspective, either this fact *is* the spectral theorem for bounded operators [RS80, p227], or else its proof follows from the spectral theorem [Jon09, p23]. Hence, von Neumann algebra theory is often called 'non-commutative measure theory'.

Morphisms and the abstract/spatial distinction

Although a von Neumann algebra M is defined as a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert $\mathcal{H}$, we make a distinction between its *abstract* and *spatial* properties. Abstract properties are recoverable from the structure of M as a normed $*$-algebra[1] – i.e., the addition, multiplication, $*$-operation, and norm $\|\cdot\|$. Spatial properties are those which depend on the underlying space $\mathcal{H}$ and the representation of M on $\mathcal{H}$. A morphism of von Neumann algebras is defined to preserve abstract but not spatial properties.

To make the definition, we recall some elementary definitions from functional analysis that carry over to von Neumann algebras. We write that $x \in M$ is *positive* if there exists $y \in M$ such that $x = y^*y$. If $x, y \in M$ are self-adjoint, then we write $x \leq y$ if $y - x$ is positive. This forms a partial order on self-adjoint elements of M. A linear map $\psi : N \to M$ is called positive if, whenever $x \in N$ is positive, $\psi(x) \in M$ is positive.

Definition 1.2.9. Suppose $N \subset \mathcal{B}(\mathcal{H})$ and $M \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras. A positive linear map $\psi : N \to M$ is *normal* if $\psi\left(\bigvee_\alpha x_\alpha\right) = \bigvee_\alpha \psi\left(x_\alpha\right)$ whenever $\{x_\alpha\}$ is an increasing, norm-bounded net of self-adjoint elements of N. (Here we have written $\bigvee_\alpha$ to mean the supremum over α.)

Definition 1.2.10. Suppose $N \subset \mathcal{B}(\mathcal{H})$ and $M \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras. A morphism of von Neumann algebras is a normal unital[2] $*$-homomorphism $\psi : N \to M$.

[1] For the reader familiar with C^*-algebras, we note that von Neumann algebras can be characterised as C^*-algebras which are Banach space duals [Sak56, p763]. This abstract characterisation makes no reference to a Hilbert space.

[2] $\psi(1) = 1$.

In particular, an inclusion of von Neumann algebras $N \subset M$ is *always* assumed to be unital, i.e. we require that $1 \in N$. We often consider an inclusion $N \subset M$ as an object in its own right. Naturally, a morphism between inclusions $N_1 \subset M_1$ and $N_2 \subset M_2$ is a morphism $\psi : M_1 \to M_2$ such that $\psi(N_1) \subset N_2$.

A bijective von Neumann algebra morphism is an isomorphism. Henceforth, we use 'abstract' to mean properties preserved under von Neumann algebra isomorphisms. We use 'abstract isomorphism' interchangeably with 'isomorphism'.

A priori, the operator topologies on a von Neumann algebra are spatial, as they are inherited from $\mathcal{B}(\mathcal{H})$ and hence depend on $\mathcal{H}$. However, we note that normality is equivalent to a continuity assumption.

Lemma 1.2.11. *A positive linear map (in particular, any $*$-homomorphism) $\psi : N \to M$ is ultraweakly continuous if and only if it is normal.*

See [Con00, 46.5]. Hence the ultraweak topology on a von Neumann algebra M is preserved under abstract isomorphism, although the other operator topologies may not be. However, this is never a problem. Almost all topological results in this thesis relate to the closure of a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, which is the same in all four operator topologies by Lemma 1.2.3. Therefore, when A is a $*$-subalgebra, the object $\overline{A}$ is well-defined under abstract isomorphisms.

We also make the notion of 'spatial property' precise by talking about representations. When we assume that a M is a von Neumann algebra, we tacitly assume $M \subset \mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$, but, because we view M as an abstract object, we do not consider $\mathcal{H}$ to be a 'privileged' representation of M. Indeed, we are frequently interested in multiple representations of a von Neumann algebra.

Definition 1.2.12. A representation of a von Neumann algebra M on a Hilbert space $\mathcal{H}$ is an von Neumann algebra morphism $\psi : M \to \mathcal{B}(\mathcal{H})$.

If ψ is a faithful (injective) representation, then M is isomorphic to its image, and we typically *identify* it with its image. Equivalence of representations is given by the notion of spatial isomorphism.

Definition 1.2.13. Suppose $\mathcal{H}, \mathcal{K}$ are Hilbert spaces and $N \subset \mathcal{B}(\mathcal{H}), M \subset \mathcal{B}(\mathcal{K})$ are von Neumann algebras. Suppose $u : \mathcal{H} \to \mathcal{K}$ is unitary and that $uNu^* = M$. Then the map $x \mapsto uxu^*$ is called a spatial isomorphism or unitary equivalence.

Henceforth, 'spatial' refers to properties depending on a choice of faithful representation, and hence preserved under spatial isomorphism. In particular, commutants are spatial, so we modify Definition 1.2.1 to clarify this.

Definition 1.2.14. If M has a faithful representation on $\mathcal{H}$, then the *commutant of M on $\mathcal{H}$* is the von Neumann algebra $\{x \in \mathcal{B}(\mathcal{H}) : xy = yx \ \forall\ y \in M\}$, which we denote by $M' \cap \mathcal{B}(\mathcal{H})$ or simply M' if the choice of representation is implicit.

E.g., if we write $M \subset \mathcal{B}(\mathcal{H})$, then it is implied M' is the commutant on $\mathcal{H}$.

1.3 Projections in a von Neumann algebra

A study of von Neumann algebras is inseparable from the study of its projections. Recall p is said to be a projection if $p = p^* = p^2$.

If M is von Neumann, then the von Neumann subalgebra $\langle x \rangle$ generated by a self-adjoint x is abelian and so isomorphic to some $L^\infty(X)$ (see Example 1.2.8). The measurable indicator functions on X map to a family of projections in M called the *spectral projections* of x. The spectral theorem implies x is contained in the strong closure of its spectral projections [RS80, pp234-235]. As $\langle x \rangle$ is strongly closed, it follows that *a von Neumann algebra is generated by its projections.* Hence, understanding projections in M is tantamount to understanding M.

Recall that self-adjoint operators have a partial order where $x \leq y$ if and only if $y - x$ is positive. Projections form a suborder. Write $p \vee q$ (resp. $p \wedge q$) for the supremum (resp. infimum) of two projections p, q with respect to the partial order. For convenience, we collect standard results about projections which we use extensively; we will not refer back to this lemma. See also [Jon09, p20].

Lemma 1.3.1. *Suppose $p, q \in \mathcal{B}(\mathcal{H})$ are projections, $x \in \mathcal{B}(\mathcal{H})$ is arbitrary, and $S \subset \mathcal{B}(\mathcal{H})$ is a set of projections.*

1. *x commutes with p if and only if $\operatorname{im} p$ is invariant under x and x^*.*
2. *$p \leq q$ if and only if $\operatorname{im} p \subset \operatorname{im} q$ if and only if $pq = p$.*
3. *$p \vee q$ is the projection onto $\operatorname{im} p + \operatorname{im} q$. Moreover, $\bigvee_{p \in S} p$ is the projection onto $\overline{\sum_{p \in S} \operatorname{im} p}$.*
4. *$p \wedge q$ is the projection onto $\operatorname{im} p \cap \operatorname{im} q$. Moreover, $\bigwedge_{p \in S} p$ is the projection onto $\bigcap_{p \in S} \operatorname{im} p$.*
5. *$p \wedge q = 0$ if and only if $pq = 0$. (If so, p, q are said to be* mutually orthogonal*.)*
6. *Suppose M is a von Neumann algebra. The set of projections in M is closed under arbitrary suprema and infima [Jon09, 20].*

Although important and used extensively, this ordering (henceforth the 'usual order') on projections reveals little about a von Neumann algebra's structure. An order that reveals more structure is the *Murray-von Neumann ordering.*

Recall that $u \in \mathcal{B}(\mathcal{H})$ is called a partial isometry if $u|_{(\ker u)^\perp}$ is an isometry. Equivalently, u is a partial isometry if u^*u and uu^* are projections. Call $\operatorname{im}(u^*u) = (\ker u)^\perp$ and $\operatorname{im}(uu^*) = \operatorname{im} u$ the initial and final spaces of u.

p, q are equivalent in the usual ordering if and only if $p = q$, or, equivalently, $\operatorname{im} p = \operatorname{im} q$. Under the Murray-von Neumann ordering, for p, q to be equivalent, it suffices for $\operatorname{im} p$ and $\operatorname{im} q$ to be related by a partial isometry.

Definition 1.3.2. (Murray-von Neumann ordering) [3] [MV36, p151]

Suppose M is a von Neumann algebra and $p, q \in M$ are projections.

1. $p \preceq q$ if there exists a partial isometry $u \in M$ with $uu^* = p$ and $u^*u \leq q$.

2. $p \approx q$ if there exists a partial isometry $u \in M$ with $uu^* = p$ and $u^*u = q$. We say p, q are *Murray-von Neumann equivalent.*

The Murray-von Neumann ordering is a partial order on the set of Murray-von Neumann equivalence classes in M; in particular if $p \preceq q$ and $q \preceq p$, then $p \approx q$. See [Jon09, 6.1.2] for a proof.

An extremely important data point for the classification of a von Neumann algebra M is its order type.

Example 1.3.3. $p \preceq q$ in $\mathcal{B}(\mathcal{H})$ if and only if $\dim(\operatorname{im} p) \leq \dim(\operatorname{im} q)$. Consequently, the order type of $\mathcal{B}(\mathcal{H})$ is the linear order $\{0, 1, \ldots, \dim \mathcal{H}\}$.

Another data point is whether the identity is equivalent to another projection.

Definition 1.3.4. A projection $p \in M$ is said to be *infinite* if $p \approx q$ for some $q < p$ (that is, q such that $q \leq p$ but $q \neq p$). Otherwise, we say p is *finite.*

Definition 1.3.5. A von Neumann algebra M is *finite* if $1 \in M$ is finite.

Example 1.3.6. $\mathcal{B}(\mathcal{H})$ is finite if and only if $\dim \mathcal{H} < \infty$. To see that $\mathcal{B}(\mathcal{H})$ is infinite when $\dim \mathcal{H} = \infty$, let $\mathcal{H}$ be $\ell^2(\mathbb{N})$, and r be the right shift operator (so r^* is the left shift). Then $r^*r = 1$ while $rr^* \neq 1$.

[3] A note on terminology: although the Murray-von Neumann order is theoretically significant, we use the usual ordering $\leq$ far more frequently. E.g. when we write 'p is a subprojection of q', 'q dominates p', etc., we mean $p \leq q$ unless otherwise specified.

We caution that finite von Neumann algebras are not literally finite sets, nor are they in general finite-dimensional.

Our final remarks on projections concern subalgebras induced by projections. Given a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ and a projection $p \in \mathcal{B}(\mathcal{H})$, form the set $pMp \subset \mathcal{B}(p\mathcal{H})$. If $x \in M$ is viewed as a block matrix subject to the decomposition $\mathcal{H} = p\mathcal{H} \oplus (1-p)\mathcal{H}$, then pxp is the top-left corner. We call pMp a cutdown of M by p.

Proposition 1.3.7. *If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $p \in M$ or $p \in M'$, then the commutant*[4] *commutes with the cutdown, i.e. $pMp = (pM'p)'$ and $(pMp)' = pM'p$. Moreover, $pMp, pM'p$ are von Neumann algebras on $\mathcal{B}(p\mathcal{H})$.*

See [Jon09, p21] for a proof. If $p \in M'$, we can write pMp as Mp. A key difference between a cutdown by $q \in M$ and by $p \in M'$ is the following result.

Lemma 1.3.8. *If M is a von Neumann algebra and $p \in M'$, then the map $x \mapsto xp$ is a von Neumann algebra morphism of M onto Mp.*

Proof. Right-multiplication by p is continuous in any of the four operator topologies [Jon09, 3.4.1] and hence normal. As p is self-adjoint and commutes with M, the map is a $*$-homomorphism. It is unital as p is the identity on $p\mathcal{H}$. $\square$

We conclude our prerequisite discussion of projections here, although of course projections will occur extensively throughout this thesis.

1.4 Factors and their classification

The central objects of this thesis are II_1 subfactors, which necessitates a discussion of *factors*.

Definition 1.4.1. A von Neumann algebra M is a factor if its centre is trivial, i.e. $Z(M) = \mathbb{C}1$. A *subfactor* is an inclusion $N \subset M$ where N, M are factors.

Factoriality is equivalent to a simplicity requirement – a von Neumann algebra is a factor if and only if it contains no strongly-closed ideals [Jon09, 6.1.12]. Thus the study of factors is as natural a subdiscipline of von Neumann algebra theory as the study of simple groups or rings is in abstract algebra.

[4] As we have mentioned, commutants are implicitly taken with respect to the space where they make sense, e.g. $M' \subset \mathcal{B}(\mathcal{H})$ whereas $(pMp)' \subset \mathcal{B}(p\mathcal{H})$.

A major achievement of Murray and von Neumann was the complete classification of factors by just two parameters: the order type of $\preceq$, and the finiteness of the factor. The key observation is the following.

Proposition 1.4.2. *The Murray-von Neumann order in a factor M is a total order on Murray-von Neumann equivalence classes. That is, for all projections $p, q \in M$, either $p \preceq q$ or $p \succeq q$. If $p \preceq q$ and $p \succeq q$, then $p \approx q$.*

See [Jon09, 6.1.8] for a proof. Hence every factor is associated to a linear order type. The classification splits factors into types I, II, III, with subtypes I_n for $n \in \mathbb{N} \cup \{\infty\}$, as well as subtypes II_1 and II_∞.

Theorem 1.4.3. *(Classification of factors) [MV36, p172]*

Suppose M is a factor. Then the Murray-von Neumann order on equivalence classes in M is isomorphic to a finite total order, or $\mathbb{N}$, or $\mathbb{R}$. Moreover, the type of M is given by the following table.

Finiteness/Order type	Finite order	$\mathbb{N}$	$\mathbb{R}$
Finite factor	I_n		II_1
Infinite factor	III	I_∞	II_∞

Specifically, a type I_n factor has order type $\{1, \ldots, n\}$ for $n \in \mathbb{N}$, and a type III factor has order type $\{0, 1\}$.

We leave the definition of a type I factor to Section 1.5, and a type II factor to Section 1.7, and not consider type III factors at all. We conclude this section with a useful lemma. Recall Proposition 1.3.7 and Lemma 1.3.8. If M is a factor, then those results can be strengthened. See [Jon09, 3.4.4] for a proof.

Lemma 1.4.4. *Suppose M is a factor.*

1. *If $p \in M$ or $p \in M'$ is a projection, then pMp is a factor.*

2. *If $p \in M'$ is a nonzero projection, then $x \mapsto xp$ is an isomorphism $M \to Mp$.*

1.5 Finite-dimensional von Neumann algebras

Although we are interested in II_1 factors and subfactors, which are necessary infinite-dimensional, tools from *finite-dimensional* von Neumann algebra theory are indispensable. Although this theory is very simple, it reappears extensively in later chapters, so we dedicate a sizable amount of space to building familiarity.

First, we describe the structure of a finite-dimensional von Neumann algebra. Any such algebra is a direct sum of finite-dimensional *type I factors*, which are isomorphic to matrix algebras.

Definition 1.5.1. If M is a von Neumann algebra, a nonzero projection $p \in M$ is said to be minimal if, for all projections $q \in M$, $q \leq p$ implies $q = p$ or $q = 0$.

Definition 1.5.2. A factor M is of type I if M contains a minimal projection, or equivalently if M is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

See [Jon09, 4.2.1] for a proof of the equivalence. One direction is easy: $\mathcal{B}(\mathcal{H})$ has trivial centre, and the projections of rank 1 in $B(\mathcal{H})$ are minimal.

Definition 1.5.3. A type I_n factor is a type I factor that is isomorphic to $M_n(\mathbb{C})$ (if n is finite) or $\mathcal{B}(\mathcal{H})$ for infinite-dimensional $\mathcal{H}$ (if $n = \infty$).

Now we decompose an arbitrary finite-dimensional von Neumann algebra M. Observe that $Z(M) \cong \mathbb{C}^{\oplus m}$ for some $m \in \mathbb{N}$. One can prove this elementarily [Jon09, 4.2.1,4.4.1], or by noting that $Z(M) \cong L^\infty(X)$, then proving X is an atomic measure space.

Let P be the set of minimal projections for $Z(M)$. These are identified with direct sums with 1 in one component and 0 in the rest. It's clear that P is a set of mutually orthogonal projections and $\sum_{p \in P} p = 1$.

Definition 1.5.4. A minimal central projection for M is a projection $p \in Z(M)$ that is minimal in $Z(M)$.

By the above remarks, we then obtain a decomposition.

Theorem 1.5.5. *Suppose M is a finite-dimensional von Neumann algebra, and P is the set of minimal central projections of M. Then, M can be written as a direct sum of finite-dimension type I factors:*

$$M = \bigoplus_{p \in P} Mp. \tag{1.1}$$

Moreover, Mp is a simple ideal for each $p \in P$.

Proof. As $\sum_{p \in P} p = 1$, the decomposition is immediate. Mp is a factor by Lemma 1.4.4. Because Mp is finite-dimensional, it contains a minimal projection. Therefore, Mp is a finite-dimensional type I factor. It follows that $Mp \cong M_{n_p}(\mathbb{C})$ for some $n_p \in \mathbb{N}$, and hence it is simple; it is an ideal because p is central in M. □

Definition 1.5.6. Suppose M is a finite-dimensional von Neumann algebra and P is its set of minimal central projections (MCPs). The *dimension vector* of M is $\vec{n}^M := (n_p^M)_{p\in P} \in \mathbb{N}^P$, where $n_p^M \in \mathbb{N}$ is the integer such that Mp is type $I_{n_p^M}$.

If P is ordered, we identify $\vec{n}^M$ with a column vector.

It follows from Definition 1.5.3 and Theorem 1.5.5 that M is (non-canonically) isomorphic to $\bigoplus_{p\in P} M_{n_p^M}(\mathbb{C})$. It is then clear that the algebraic data of a finite-dimensional M is *completely* encoded by the vector $\vec{n}^M$.

Finite-dimensional inclusions

Besides lone von Neumann algebras, we extensively deal with inclusions $N \subset M$. Whilst a vector encodes a lone algebra, a matrix encodes an inclusion.

Theorem 1.5.7. *Suppose $N \subset M$ is an inclusion of finite-dimensional von Neumann algebras, where P (resp. Q) is the set of minimal central projections of N (resp. M). Let $\iota : N \to M$ denote the inclusion map. There exist unique integers $\lambda_{pq} \in \mathbb{N}$ for $p \in P, q \in Q$, and a choice of isomorphisms of N onto $\bigoplus_{p\in P} M_{n_p^N}(\mathbb{C})$ and M onto $\bigoplus_{q\in Q} M_{n_q^M}(\mathbb{C})$ such that $\iota : N \to M$ is given by:*

$$x = \bigoplus_{p\in P} x_p \longmapsto \bigoplus_{q\in Q} \left(\bigoplus_{p\in P} x_p^{\oplus\lambda_{pq}} \right).$$

Definition 1.5.8. $\Lambda_N^M := (\lambda_{pq})_{p\in P, q\in Q}$ is the *inclusion matrix* of the pair $N \subset M$, where the rows are indexed by the set P and the columns are indexed by Q.

See [Jon09, pp28-29] for a proof. The result is best illustrated by example.

Example 1.5.9. Suppose that $\vec{n}^N = (1, 2, 4)$ and $\vec{n}^M = (5, 8)$. By Definition 1.5.6, this means that $N \cong M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_4(\mathbb{C})$, and $M \cong M_5(\mathbb{C}) \oplus M_8(\mathbb{C})$. Suppose the inclusion matrix of $N \subset M$ is given by (1.2). Then Theorem 1.5.7 states that the inclusion $\iota : N \to M$ is identified with the map of (1.3).

$$\Lambda_N^M = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{1.2}$$

$$\iota : (a \oplus b \oplus c) \mapsto \begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix} \oplus \begin{pmatrix} b & & \\ & b & \\ & & c \end{pmatrix} \tag{1.3}$$

where $a \in M_1(\mathbb{C}), b \in M_2(\mathbb{C}), c \in M_4(\mathbb{C})$. (Empty entries are zero.)

The (p, q)th entry of the inclusion matrix (i.e. λ_{pq}) counts the 'number of copies' of the Np appearing in Mq. For example, in (1.2), we see that $\lambda_{2,1} = 1$ and $\lambda_{2,2} = 2$, which corresponds to the one copy of b and two copies of b in the two components of (1.3), respectively. We can recast this information as a graph.

Definition 1.5.10. Suppose P (respectively Q) is the set of minimal central projections of N (respectively M). Then the *Bratteli diagram* of the inclusion $N \subset M$, denoted by β_N^M, is the bipartite graph[5] with left vertices P, right vertices Q, and λ_{pq} edges drawn from p to q.

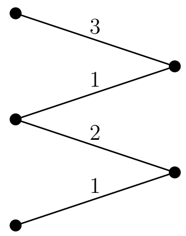


Figure 1.1: The Bratteli diagram associated to the inclusion map (1.3).

In graph theory, Λ_N^M is called the *biadgacency* matrix[6] of β_N^M. As a bipartite graph is uniquely specified by its biadjacency matrix, *either* the pair $\left(\Lambda_N^M, \vec{n}^N\right)$ *or* the pair $\left(\beta_N^M, \vec{n}^N\right)$ encodes all the algebraic data of $N \subset M$. (The vector $\vec{n}^M$ is unneeded as it can be recovered using Lemma 1.5.11.)

These objects encode constraints on the inclusion map. Inclusions of von Neumann algebras are by definition unital. This means that the inclusion map must 'fill' all diagonals of M. As inclusions are also injective, every component of N must appear in at least one component of M. The following map fails to be an inclusion map on both counts:

$$\iota : (a \oplus b \oplus c) \mapsto \begin{pmatrix} a & & \\ & b & \\ & & 0 \end{pmatrix} \begin{pmatrix} b & & & \\ & b & & \\ & & b & \\ & & & b \end{pmatrix}$$

By counting dimensions, unitality implies $\sum_{p \in P} n_p^N \lambda_{pq} = n_q^M$, i.e. $\vec{n}^N \cdot \Lambda_N^M = \vec{n}^M$. In particular, Λ_N^M cannot have a column of zeroes. Injectivity implies every row of Λ_N^M is nonzero. This translates to the following pair of constraints on the inclusion matrix and Bratteli diagram.

[5]Henceforth 'graph' refers to an undirected multigraph (i.e., multiple edges are allowed).
[6]Not to be confused with the adjacency matrix.

Lemma 1.5.11. *The inclusion matrix* Λ_N^M *has no trivial rows or columns. Moreover,* $\vec{n}^N \cdot \Lambda_N^M = \vec{n}^M$.

Lemma 1.5.12. *The Bratteli diagram* β_N^M *has no isolated vertices.*

The inclusion matrix allows us to visualise the relationship between $N \subset M$ and the 'dual inclusion' $M' \subset N'$.

Proposition 1.5.13. *Suppose* $N \subset M$ *is an inclusion of finite-dimensional von Neumann algebras on* $\mathcal{H}$. *Then* $\Lambda_{M'}^{N'} = (\Lambda_N^M)^T$.

Example 1.5.14. We omit the proof and give an example. Let $\vec{n}^N = (2,4)$, $\vec{n}^M = (14)$, and let Λ_N^M be the 2×1 matrix $(3,2)$. The commutant depends on the space M acts on[7]; assume M acts on $\mathbb{C}^{14}$ so we can identify M spatially with $M_{14}(\mathbb{C})$. Then an arbitrary element $x \in N$ is given by

$$x = \begin{pmatrix} a & & & & \\ & a & & & \\ & & a & & \\ & & & b & \\ & & & & b \end{pmatrix} = \begin{pmatrix} a \otimes I_3 & \\ & b \otimes I_2 \end{pmatrix}$$

where $a \in M_2(\mathbb{C}), b \in M_4(\mathbb{C})$. Then an arbitrary element $y \in N'$ is given as follows, where $c \in M_3(\mathbb{C}), d \in M_2(\mathbb{C})$.

$$y = \begin{pmatrix} I_2 \otimes c & \\ & I_4 \otimes d \end{pmatrix}$$

Hence $N' \cong M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$. As $M = M_{14}(\mathbb{C})$, its commutant on $\mathbb{C}^{14}$ is the scalar multiples of the identity, i.e. $M' \cong M_1(\mathbb{C})$. Hence three copies of M' are included in the first simple summand of N', and two copies are included in the second summand. That is, $\Lambda_{M'}^{N'} = (3,2)^T$, as required.

We will use these combinatorial facts frequently. To summarise: a finite-dimensional von Neumann algebra N is fully described by $\vec{n}^N$ whereas an inclusion $N \subset M$ is fully described either by $(\vec{n}^N, \Lambda_N^M)$ or $(\vec{n}^N, \beta_N^M)$

1.6 Tracial von Neumann algebras

The remainder of this chapter has more direct importance to subfactor theory than the preceding sections. In particular, the material of this section (Section 1.6) will be used extremely extensively.

[7] In general M can act on $\mathbb{C}^{14k}$ for $k \in \mathbb{N}$, in which case we would need to identify M spatially with $M_{14}(\mathbb{C}) \otimes I_k$.

The next family of von Neumann algebras that we consider are von Neumann algebras equipped with a (nice) trace, of which II_1 factors are a subfamily. The presence of a trace confers an enormous amount of extra structure.

Definition 1.6.1. Suppose M is a von Neumann algebra and $\phi : M \to \mathbb{C}$ is a linear functional.

- ϕ is positive if $\phi(x^*x) \geq 0$ and $\phi(x^*) = \overline{\phi(x)}$ for all $x \in M$.
- ϕ is faithful if $\phi(x^*x) = 0 \implies x = 0$ for all $x \in M$.
- ϕ is normalised if $\phi(1) = 1$.
- ϕ is a trace if $\phi(xy) = \phi(yx)$ for all $x, y \in M$.

Definition 1.6.2. A *tracial von Neumann algebra* is a pair (M, tr) where M is a von Neumann algebra and $\mathrm{tr} : M \to \mathbb{C}$ is a positive faithful normal[8] normalised trace.

If (M, tr) is a tracial von Neumann algebra, we typically leave the trace implicit by writing "M is a tracial von Neumann algebra". However, despite this abbreviation, the trace is considered part of the data of the object. To refer to the trace on M that was left implicit, we will write tr_M.

We require that morphisms of tracial von Neumann algebras preserve the trace. In particular, inclusions must be trace-preserving.

Definition 1.6.3. A *tracial* or *trace-preserving inclusion* is a pair $(N \subset M, \mathrm{tr})$, where $N \subset M$ is an inclusion of von Neumann algebras, and $\mathrm{tr} : M \to \mathbb{C}$ is a positive faithful normal normalised trace.

Note that a tracial inclusion is of course equivalent to a pair of tracial von Neumann algebras $(N, \mathrm{tr}_N), (M, \mathrm{tr}_M)$ such that $N \subset M$ and tr_M extends tr_N. We will study tracial inclusions extensively throughout the thesis, as II_1 subfactors are examples of tracial inclusions. (See Definition 1.8.1.)

Standard form of a tracial von Neumann algebra

Traciality is a powerful condition. Chiefly, it unlocks the *GNS construction*[9], a method to build a *standard form representation* for a tracial von Neumann

[8]Equivalently, ultraweakly continuous, by Lemma 1.2.11.

[9]We present a special case of the GNS construction that is relevant to this thesis. The general GNS construction takes as input (M, ϕ), where $\phi : M \to \mathbb{C}$ only needs to be positive and normalised.

algebra. For an analogy, consider the abelian von Neumann algebra $L^\infty(X,\mu)$. The algebra $L^\infty(X,\mu)$ has a canonical action on the *vector space* $L^\infty(X,\mu)$ by left multiplication, i.e. $f \cdot g := fg$. Moreover, a natural inner product on the vector space $L^\infty(X,\mu)$ is the L^2 inner product $\langle f, g\rangle_{L^2} = \int_X \overline{g} f d\mu$.

This generalises to an arbitrary tracial von Neumann algebra (M, tr), where the trace tr plays the role of the integral $\int d\mu$. Let $\hat{M} = \{\hat{x} : x \in M\}$ denote M viewed as a vector space, and have M act on $\hat{M}$ by left multiplication, i.e. $a \cdot \hat{b} := \widehat{ab}$. Define an inner product on $\hat{M}$ as follows:

Definition 1.6.4. Let $\langle\cdot,\cdot\rangle_{L^2} : \hat{M} \times \hat{M} \to \mathbb{C}$ be defined by $\langle \hat{x}, \hat{y}\rangle_{L^2} := \mathrm{tr}(y^* x)$, and $\|\hat{x}\|_{L^2} := \langle x, x\rangle_{L^2}^{1/2}$ for all $x \in M$. Let $L^2(M, \mathrm{tr})$ denote the completion of $\hat{M}$ with respect to $\langle\cdot,\cdot\rangle_{L^2}$.

The form $\langle\cdot,\cdot\rangle_{L^2}$ is an inner product as tr is positive and faithful. We abbreviate $L^2(M,\mathrm{tr})$ to $L^2(M)$, but we are careful to remember that $L^2(M)$ depends on the choice of trace.

A fact in von Neumann algebra theory is that the Hölder-like inequality $\|\widehat{xy}\|_{L^2} \le \|x\| \|\hat{y}\|_{L^2}$ holds for any x, y in a tracial von Neumann algebra [Jon09, 6.2.6]. Note here $\|x\|$ denotes the operator norm of x, and not an L^2 norm. This means the action of any element x on $\hat{M}$, $\hat{y} \mapsto \widehat{xy}$, is always bounded, and hence extends to a unique bounded operator on $L^2(M)$. This gives a canonical representation of M.

Theorem 1.6.5. *(Standard form)*

Suppose M is a tracial von Neumann algebra. Then M has a unique faithful representation on $L^2(M)$ such that M acts on the dense subspace $\hat{M}$ by left multiplication, i.e. $x \cdot \hat{y} = \widehat{xy}$ for all $\hat{y} \in \hat{M}$.

Moroever, the trace on M is given by $\mathrm{tr}_M(x) = \langle x\hat{1}, \hat{1}\rangle_{L^2}$.

See [Arv76, p27-30] [Tak02, 3.12]. (The last assertion is easy.) The standard form of M is easy to work with, as we can explicitly describe the action of any $x \in M$ on the dense subspace $\hat{M}$. Henceforth, unless we specify a choice of representation for a tracial von Neumann algebra M, we imply that M is identified with a subalgebra of $\mathcal{B}(L^2(M))$.

Properties of the standard form

We summarise the key structures of the standard form, which we use extensively.

Theorem 1.6.5 uses all of the hypotheses on tr, except for the fact that is a trace. However, the fact that tr is a trace equips $L^2(M)$ with a key piece of extra structure. Consider the $*$-operation on M, viewed as an antilinear map $\hat{M} \to \hat{M}$ sending $\hat{x}$ to $\widehat{(x^*)}$. Becaue tr is cyclic, $\|\hat{x}\|_{L^2}^2 = \mathrm{tr}(x^*x) = \mathrm{tr}(xx^*) = \|\widehat{x^*}\|_{L^2}^2$, and hence the $*$-operation is an isometry in the L^2-norm. Hence it extends to $L^2(M)$.

Definition 1.6.6. $J : L^2(M) \to L^2(M)$ denotes the unique antiunitary involution such that $J\hat{x} = \widehat{(x^*)}$ for all $x \in \hat{M}$.

Lemma 1.6.7. *Conjugation by J, denoted by* $\mathrm{Ad}\, J : x \mapsto JxJ$*, is an injective antilinear von Neumann algebra morphism.*

The proof is easy: one readily checks that $\mathrm{Ad}\, J$ respects addition, multiplication, and adjoints, but conjugates scalars. The existence of this map means M not only acts on $L^2(M)$ by left multiplication, but also right multiplication.

Proposition 1.6.8. *For all $x \in M$, there exists a unique bounded operator on $L^2(M)$ extending the map $\hat{y} \mapsto \widehat{yx}$, and it is given by Jx^*J.*

Proof. Jx^*J is bounded because J, x are. To see that it is the requisite map, observe that $Jx^*J\hat{y} = Jx^*\widehat{(y^*)} = J\widehat{(x^*y^*)} = \widehat{yx}$. □

We call the operator extending $\hat{y} \mapsto \widehat{yx}$ the right-multiplication operator by x, and write $\xi \cdot x$ to represent it acting on $\xi \in L^2(M)$. This leads to an explicit expression for the commutant of M on $L^2(M)$.

Lemma 1.6.9. *$JMJ = M'$, where M' is the commutant of M on $L^2(M)$.*

Right-multiplication commutes with left-multiplication, so $JMJ \subset M'$. Proving the reverse inclusion amounts to proving that *every* element of the commutant M' is given by a right-multiplication operator; we defer it to [Jon09, 9.1.6].

Another key property of the standard form is that an inclusion of tracial von Neumann algebras extends to an inclusion of L^2 spaces.

Proposition 1.6.10. *If $N \subset M$ is an inclusion of tracial von Neumann algebras, then the inclusion map $\iota : \hat{N} \to \hat{M}$ is an isometry, and extends to an isometry $\iota : L^2(N) \to L^2(M)$.*

Proof. The inclusion is trace-preserving by Definition 1.6.2. Hence $\iota : \hat{N} \to \hat{M}$ is an isometry for $\|\cdot\|_{L^2}$, and so extends to an isometry $L^2(N) \to L^2(M)$. □

We can therefore write $L^2(N) \subset L^2(M)$.

Our final result is a way to intrinsically identify the dense subspace $\hat{M}$ inside $L^2(M)$. For any $\xi \in L^2(M)$, define a right-multiplication map $R_\xi : \hat{M} \to L^2(M)$

by $R_\xi(\hat{y}) = y\xi$. We say ξ is a right-bounded vector if R_ξ is bounded with respect to the norm $\|\cdot\|_{L^2}$ on both spaces. Equivalently, ξ is right-bounded if R_ξ extends to an operator in $\mathcal{B}(L^2(M))$. This gives our characterisation of $\hat{M}$.

Proposition 1.6.11. *$\hat{M}$ equals the set of right-bounded vectors in $L^2(M)$.*

Proof. If $\hat{x} \in M$, then $R_{\hat{x}}$ is the right-multiplication operator by x, which is bounded by Proposition 1.6.8. If ξ is a right-bounded vector, then $\xi \in \hat{M}$ if and only if R_ξ equals a right-multiplication operator by some element of M, if and only if $R_\xi \in JMJ = M'$. For all $\hat{z} \in \hat{M}$, $R_\xi(y\hat{z}) = R_\xi(\widehat{yz}) = yz\xi = y(R_\xi\hat{z})$. Hence R_ξ commutes with M on the dense subspace $\hat{M}$. □

We thus have an ample selection of tools with which to probe a tracial von Neumann algebra M, so long as we represent it in standard form on $L^2(M)$.

1.7 II_1 factors

We have given an overview of several important families of von Neumann algebras; we now specialise to the specific family of interest to this thesis. Recall that the Murray-von Neumann classification of factors (Theorem 1.4.3) characterises II_1 factors as finite factors with Murray-von Neumann order type $[0, 1]$. However, they have an equivalent, seemingly unrelated definition.

Definition 1.7.1. A II_1 factor M is an infinite-dimensional tracial factor.

A priori, this means that a II_1 factor is a pair (M, tr) where tr is a positive faithful normal normalised trace. However, M has a unique such trace.

Theorem 1.7.2. *Suppose (M, tr) is a II_1 factor and $\widetilde{\mathrm{tr}}$ is a normal normalised trace. Then $\mathrm{tr} = \widetilde{\mathrm{tr}}$.*

That is, the structure of a II_1 factor *as a tracial object* is determined by its underlying structure *as a von Neumann algebra*. See [Tak02, V.2.6] for a proof.

This is an example of what makes II_1 factors so interesting – not only do they inherit the considerable structure of factors (described in Section 1.4) and tracial von Neumann algebras (Section 1.6), but the structures interact in powerful ways. To see another example, recall from Proposition 1.4.2 that the Murray-von Neumann equivalence classes in a factor are totally ordered. In fact, the order in a II_1 factor is specified by the trace.

Proposition 1.7.3. *Suppose M is a II_1 factor. Then* $\mathrm{tr}(p) \leq \mathrm{tr}(q) \iff p \preceq q$. *Moreover,* $\{\mathrm{tr}(p) : p \in M$ *is a projection.*$\} = [0,1]$.

This means that, indeed as stated in the classification, a II_1 factor has Murray-von Neumann order type $[0,1]$. Notice the similarity of II_1 factors to $L^\infty([0,1])$ (equipped with the trace $f \mapsto \int_0^1 f dx$). A II_1 factor and $L^\infty([0,1])$ both contain projections of arbitrary real-valued trace in $[0,1]$.

The coupling constant

One of Vaughan Jones's innovations in [Jon83] was the *index*, a measure of relative size for a II_1 factor inside another II_1 factor. He leveraged an existing measure of size introduced by Murray and von Neumann: the *coupling constant* $\dim_M \mathcal{H}$, which measures the size of a representation of M on $\mathcal{H}$. To define this, we need a common setting in which to compare representations of M. Note that representations of factors are faithful, as factors have no strongly-closed ideals.

By virtue of being tracial, a II_1 factor has a standard form representation on $L^2(M)$ (Theorem 1.6.5). Consider the 'amplified' space $L^2(M) \otimes \ell^2(\mathbb{N})$, where the representation of M on $L^2(M) \otimes \ell^2(\mathbb{N})$ is the map $M \to \mathcal{B}(L^2(M) \otimes \ell^2(\mathbb{N}))$ given by $x \mapsto x \otimes 1$. The amplification of $L^2(M)$ is 'large enough' to contain any representation of M as a subrepresentation.

Lemma 1.7.4. *Suppose a II_1 factor M has a representation on $\mathcal{H}$. There exists an isometry $u : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$ such that $u(x \cdot \xi) = x \cdot (u\xi)$ for all $\xi \in \mathcal{H}$.*

We say that a map u satisfying the properties given above is an *M-intertwining isometry.* See [Jon09, 10.1.1] for a proof of the above. Note u is a unitary of $\mathcal{H}$ onto $\operatorname{im} u$, and uu^* is the projection onto $\operatorname{im} u$. Hence Lemma 1.7.4 implies that M acting on $\mathcal{H}$ is spatially isomorphic to $(uu^*)(M \otimes 1)$ acting on $\operatorname{im} u$. Informally, we measure the $\mathcal{H}$-representation's 'size' by the 'size' of the subspace $\operatorname{im} u$, where 'size' is represented by a trace of uu^*. We sketch the construction of this trace.

As u commutes with the M-action, so must uu^*, and so $uu^* \in (M \otimes 1)'$. One can show that$(M \otimes 1)'$ is a II_∞ factor. II_∞ factors are distinguished from II_1 factors by having traces that are not globally defined and can take the value $+\infty$ [Jon09, p58]. We will not define or work with II_∞ theory in any generality; instead, we sketch an *ad hoc* argument that $(M \otimes 1)'$ has a trace. First note that the commutant distributes over the tensor product as follows:

$$(M \otimes 1)' = M' \otimes \mathcal{B}(\ell^2(\mathbb{N}))$$

where M' here is the commutant of M on $L^2(M)$. But the commutant of M in standard form on $L^2(M)$ is JMJ (Lemma 1.6.9), so we have that

$$(M \otimes 1)' = JMJ \otimes \mathcal{B}(\ell^2(\mathbb{N})).$$

An element $x \in JMJ \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ is represented as an infinite matrix $x = (Jx_{ij}J)_{i,j\in\mathbb{N}}$, for $x_{ij} \in M$. We attempt to define $\operatorname{tr}_{(M\otimes 1)'}(x) := \sum_{i=1}^{\infty} \operatorname{tr}_M(x_{ii})$, where tr_M is the trace on M. This sum converges in $[0, \infty]$, and is independent of the choice of matrix representation, when x is positive. This includes when x is a projection [Jon09, p58] [Jon09, 9.1.11].

Then $\operatorname{tr}(uu^*)$ is well-defined, and hence we use it to measure the 'size' of the $\mathcal{H}$-representation. This turns out to be independent of the construction of u. If $v : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$ is another M-intertwining isometry, then note that $u^*u = 1 = v^*v$ in $\mathcal{B}(\mathcal{H})$. Then $\operatorname{tr}(uu^*) = \operatorname{tr}((uv^*)(vu^*)) = \operatorname{tr}((vu^*)(uv^*)) = \operatorname{tr}(vv^*)$.

Definition 1.7.5. Suppose a II_1 factor M has a representation on $\mathcal{H}$. Then the *coupling constant* of this representation is $\dim_M(\mathcal{H}) := \operatorname{tr}(uu^*) \in (0, \infty]$, where $u : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$ is any M-intertwining isometry.

Despite the notation, the coupling constant is an invariant of the representation and not of $\mathcal{H}$. In particular, it is invariant under spatial isomorphisms.

Example 1.7.6. $\dim_M \left(L^2(M) \otimes \ell^2(\mathbb{N})\right) = \infty$ and $\dim_M L^2(M) = 1$.

We establish some important properties of the coupling constant, in order to set up the properties of the index. First, we note that the coupling constant detects the finiteness of the commutant (recall Definition 1.3.5).

Proposition 1.7.7. *If M acts on $\mathcal{H}$, then $\dim_M \mathcal{H} < \infty$ if and only if M' (the commutant of M on $\mathcal{H}$) is finite, if and only if M' is a II_1 factor.*

A proof is found in [Jon09, 9.1.9]. Heuristically, if $\mathcal{H}$ is 'larger', there are 'more' operators on $\mathcal{H}$ and hence 'more' operators in the commutant.

We also state the following without proof; see [Jon09, 10.2.1]. To interpret the first two results: if M has a representation on $\mathcal{H}$, then clearly pMp has a representation on $p\mathcal{H}$ given by restricting each $x \in M$ to $p\mathcal{H}$.

Proposition 1.7.8. *Suppose M is a II_1 factor with a representation on $\mathcal{H}$. Then,*

1. *If $p \in M$, then $\dim_{pMp}(p\mathcal{H}) = \operatorname{tr}_M(p)^{-1} \dim_M(\mathcal{H})$.*
2. *If $\dim_M \mathcal{H} < \infty$ and $p \in M'$, then $\dim_{Mp}(p\mathcal{H}) = \operatorname{tr}_{M'}(p) \dim_M \mathcal{H}$*
3. *If $\dim_M \mathcal{H} < \infty$, then $(\dim_M \mathcal{H})(\dim_{M'} \mathcal{H}) = 1$.*

This is enough background for us to move on to the theory of II_1 *subfactors*.

1.8 II_1 subfactors

Definition 1.8.1. A II_1 subfactor is an inclusion of II_1 factors $N \subset M$.

A priori, since a II_1 factor is a tracial object, we should require a II_1 subfactor to be a *tracial inclusion* (Definition 1.6.3). However, due to the uniqueness of the trace (Theorem 1.7.2), any inclusion of II_1 factors is trace-preserving.

As factors have no strongly-closed ideals, a morphism between factors is injective. Hence II_1 subfactor theory is really the theory of morphisms of II_1 factors.

Althought II_1 factor theory originated with Murray and von Neumann, the study of II_1 subfactors was not well systematised until Vaughan Jones's groundbreaking paper 'Index for Subfactors' [Jon83]. In it, Jones introduced the first major invariant for a II_1 subfactor $N \subset M$, called the *index* and denoted by $[M : N]$. In the same paper, Jones proved the first classification theorem for II_1 subfactors, using the index – a proof which we will revisit with modern techniques. Therefore, our account of modern II_1 subfactor theory must begin with the definition of the index.

The index

The notation $[M : N]$ is chosen by analogy to the group-theoretic index $[G : H]$ of a subgroup $H \subset G$ or the field-theoretic degree $[E : F]$ of a field extension E/F. Recall that the degree of E/F is defined by $[E : F] := \dim_F E$, where E is viewed an F-vector space. By restricting the standard representation of M on $L^2(M)$ to a representation of N, we can make an analogous definition:

Definition 1.8.2. The *index* of $N \subset M$ is $[M : N] := \dim_N L^2(M)$.

The index has a number of important properties, some of which resemble that of the group-theoretic index or field-theoretic degree.

Proposition 1.8.3. *[Jon83, 2.1.8,2.2.1]*

Suppose $N \subset M$ is a II_1 subfactor.

1. $[M : M] = 1$ *and* $[M : N] \in [1, \infty)$.

2. $[M : N] < \infty$ *if and only if N', the commutant of N on $L^2(M)$, is finite.*

3. *If $\mathcal{H}$ is any representation of M with* $\dim_M(\mathcal{H}) < \infty$, *then* $[M : N] = \dim_N(\mathcal{H})/\dim_M(\mathcal{H})$.

4. *If* $p \in N' \cap M$ *is a projection, then* $[pMp : Np] = [M : N] \operatorname{tr}_M(p) \operatorname{tr}_{N'}(p)$.

5. *If* $[M : N] < \infty$*, then* $[M : N] = [N' : M']$.

6. *If* $P \subset N \subset M \subset$ *are* II_1 *factors, then* $[M : P] = [M : N][N : P]$.

Proof. As $L^2(N) \subset L^2(M)$, it is easy to show that the index $[M : N] = \dim_N L^2(M) \geq \dim_N L^2(N) = 1$, proving part 1. Part 2 is immediate from Proposition 1.7.7. For part 3, see [Jon83, 2.1.7] or [Jon09, 19.2.3]. We sketch the idea. Note the identity reduces to Definition 1.8.2 when $\mathcal{H} = L^2(M)$. If $\mathcal{H}$ is another representation of M, one can show it differs from $L^2(M)$ by an amplification (tensoring by $\ell^2(\mathbb{N})$) and a cutdown. Both leave the the ratio $\dim_N(\mathcal{H})/\dim_M(\mathcal{H})$ invariant. Parts 4,5, and 6 are then straightforward calculations using part 3 and Proposition 1.7.8. □

This concludes our elementary study of the index. In the following chapter, we will prove a major theorem about it.

1.9 Conclusion

This concludes the preliminary knowledge required for our study of II_1 subfactor theory. We briefly highlight the most important parts. The combinatorial description of finite-dimensional inclusions $N \subset M$, using inclusion matrices and Bratteli diagrams, will be used frequently. Also crucial is the standard form of M on $L^2(M)$, which we use to define the key technique of [Jon83], the *basic construction*. Finally, the facts about the coupling constant and index (Proposition 1.7.8 and Proposition 1.8.3) are very important.

So far, we have only developed one theoretical tool which is specific to subfactor theory (the index). In the following chapter, we will develop many more techniques, in order to prove the Jones index theorem.

Our account of II_1 subfactor theory now begins in earnest.

Chapter 2

The basic construction and index theorem

2.1 Introduction

We are equipped to state the first major theorem in modern II_1 subfactor theory. In the same paper [Jon83] where Jones introduced the index, he determined the set of values that it can take. It is astonishing that the index can take any value above 4, but the set of indices below 4 is a discrete set.

Theorem 2.1.1. *(Jones index theorem) [Jon83, 4.3.1,4.3.2]*

$$\{[M:N] : N \subset M \textit{ is a } II_1 \textit{ subfactor.}\} = \{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty]. \quad (2.1)$$

n	$4\cos^2(\pi/n)$
3	1
4	2
5	$\left(\frac{1+\sqrt{5}}{2}\right)^2$ (Square of golden ratio)

n	$4\cos^2(\pi/n)$
6	3
7	$2 + 2\cos(2\pi/7)$
8	$2 + \sqrt{2}$ (Silver ratio + 1)

Table 2.1: The six smallest index values in the 'discrete series'.

Jones proved that the index of any subfactor is constrained to these values, and moreover constructed a II_1 subfactor for every index value in (2.1). Both directions of the proof rely on a theoretical tool introduced by Jones in the same paper [Jon83, pp7-8]: the basic construction. The basic construction is exceptionally important to subfactor theory; it has become one of the central tools of the field.

In this chapter, we introduce the basic construction, and leverage it to prove both directions of the index theorem.

If $N \subset M$ is an inclusion of tracial von Neumann algebras (henceforth 'tracial inclusion' or 'trace-preserving inclusion'), the basic construction is a procedure that adjoins a projection e_N to M to form a new von Neumann algebra $M_1 = \langle M, e_N \rangle$. In some circumstances, the basic construction can be iterated to produce a tower of algebras $N \subset M \subset M_1 \subset M_2 \subset \ldots$. Called the Jones tower, *if it exists and is unique*, then it is the ideal setting for studying $N \subset M$. We state a theorem (the tower-building theorem, Theorem 2.3.7) that unifies approaches to proving existence and uniqueness of Jones towers.

A Jones tower contains a family of projections $\{e_i\}$ satisfying what are now called the Jones relations (also called Temperley-Lieb relations). These relations have considerable emergent structure, so much that they spawned an independent field of study outside II_1 subfactor theory – the theory of *Temperley-Lieb algebras.* Temperley-Lieb algebras are signficant as they are isomorphic to *diagram algebras*, allowing elegant pictorial reasoning to substitute for algebraic tools.

We develop some pictorial tools and leverage them to prove the index theorem. We present the perspective that the index theorem is not wholly a fact in subfactor theory, but is strongly connected to Temperley-Lieb algebra theory.

2.2 The basic construction

This section draws on [Jon83], [Pop90], and [JS97].

Although the basic construction is a widely-used tool for working with II_1 subfactors, it applies more generally to any tracial inclusion (and we will need this level of generality to prove the index theorem). In this section, assume that $(N \subset M, \mathrm{tr})$ is a tracial inclusion of von Neumann algebras (see Definition 1.6.2). As usual we abbreviate this object to $N \subset M$, but we consider tr part of its data.

Identify M as a von Neumann subalgebra of $\mathcal{B}(L^2(M))$ acting by left multiplication (Theorem 1.6.5). By Proposition 1.6.10, $L^2(N) \subset L^2(M)$, and hence there exists a projection $e_N : L^2(M) \to L^2(N)$, called a Jones projection. We adjoin e_N to M to form a new von Neumann algebra.

Definition 2.2.1. The *basic construction* of a tracial inclusion $N \subset M$ is $\langle M, e_N \rangle$, the von Neumann algebra in $\mathcal{B}(L^2(M))$ generated by $M \cup \{e_N\}$.

That is, $\langle M, e_N \rangle = (M \cup \{e_N\})''$. We write $N \subset M \subset^{e_N} \langle M, e_N \rangle$ to indicate that these three algebras are a 'basic construction triplet'. The superscript e_N

indicates that e_N is an important part of the data.

To understand $\langle M, e_N\rangle$, we must first understand e_N. *A priori*, e_N maps $L^2(M)$ onto $L^2(N)$, but we can say more. It maps the subspace of $L^2(M)$ identified with M to the subspace of $L^2(N)$ identified with N.

Proposition 2.2.2. $e_N\hat{M} = \hat{N}$.

Proof. e_N acts as the identity on $\hat{N}$, so obviously $\hat{N} \subset e_N\hat{M}$.

Obviously $e_N\hat{M} \subset L^2(N)$. Now let $\hat{x} \in \hat{M}$. By Proposition 1.6.11, to show $e_N\hat{x} \subset \hat{N}$, we can equivalently show that $e_N\hat{x}$ is a right-bounded vector in $L^2(N)$. That is, we'll show $R_{e_N\hat{x}} : \hat{N} \to L^2(N)$ is bounded. Let $\hat{y} \in \hat{N}$ be arbitrary.

$$\|R_{e_N\hat{x}}\hat{y}\|_{L^2} = \|ye_N\hat{x}\|_{L^2} = \|e_N y\hat{x}\|_{L^2} = \|e_N\widehat{yx}\|_{L^2}.$$

The second equality holds as $L^2(N)$ is y-invariant, and so y commutes with e_N.

$$\|R_{e_N\hat{x}}\hat{y}\|_{L^2} \leq \|\widehat{yx}\|_{L^2} = \|(Jx^*J)\hat{y}\|_{L^2} \leq \|x\|\|\hat{y}\|_{L^2}$$

where we have used the fact that Jx^*J is the right-multiplication operator by x, and is bounded with the same norm as x, by Proposition 1.6.8. Hence $R_{e_N\hat{x}}$ is bounded on $L^2(N)$, and so $e_N\hat{x}$ is a right-bounded vector in $L^2(N)$, which implies $e_N\hat{x} \in \hat{N}$. Therefore $e_N\hat{M} \subset \hat{N}$. □

Hence, e_N can be be viewed as a map $M \to N$, which we distinguish from e_N by denoting it as E_N.

Definition 2.2.3. If $N \subset M$ is an inclusion of tracial von Neumann algebras, define $E_N : M \to N$ by the equation $\widehat{E_N(x)} = e_N\hat{x}$ for all $x \in M$.

It is clear that E_N is linear and acts as the identity on N, but otherwise its properties are not obvious.

In fact $E_N : M \to N$ is a tr-*preserving conditional expectation.* Conditional expectations are a fundamental concept in operator algebras. Given von Neumann algebras $A \subset B$, a conditional expectation $E : B \to A$ is a map sending $x \in B$ to its 'best approximation' in A.

The prototypical example is the conditional expectation in probability theory. Let $X : \Omega \to \mathbb{R}$ be a random variable on a probability space Ω with probability measure $\mathbb{P}$. The conditional expectation of X, conditioned on a subalgebra $A \subset L^\infty(\Omega)$, is a random variable denoted by $\mathbb{E}[X|A]$. It represents the best approximation to X provided we can measure variables in A but nothing else. For example, if A is singly-generated by Y, then $\mathbb{E}[X|A] = \sum_{y\in \operatorname{im} Y} \frac{1_{Y^{-1}(y)}}{\mathbb{P}(Y=y)} \int_{Y^{-1}(y)} X d\mathbb{P}$.

It is straightforward to show that $\mathbb{E}[YX|A] = Y\mathbb{E}[X|A]$. More generally, $\mathbb{E}[\cdot|\cdot]$ is defined so that, even if A isn't singly generated, this identity holds for all $Y \in A$. The conditional expectation in operator algebra theory generalises this identity.

Definition 2.2.4. A conditional expectation of M onto N is a positive unital linear map $E : M \to N$ that additionally satisfies the *bimodule property*

$$E(axb) = aE(x)b \text{ for all } x \in M,\ a, b \in N.$$

However, the map $\mathbb{E}[\cdot|\cdot]$ in probability theory has more structure than is captured by Definition 2.2.4. It depends on the choice of integral $\int_\Omega d\mathbb{P} : L^\infty(\Omega) \to \mathbb{C}$. It is also compatible with the integral; it's either in the definition of $\mathbb{E}[\cdot|\cdot]$ or a consequence of its construction that $\int_\Omega \mathbb{E}[X|A]Y d\mathbb{P} = \int_\Omega XY d\mathbb{P}$ for all $Y \in A$. To generalise this integral-compatibility to a von Neumann algebra, replace the integral with the trace.

Definition 2.2.5. A conditional expectation $E : M \to N$ is tr-*preserving* if, for all $x \in M$, $y \in N$,

$$\operatorname{tr}(E(x)y) = \operatorname{tr}(xy). \tag{2.2}$$

We can now prove our claim about E_N.

Theorem 2.2.6. *$E_N : M \to N$ is a* tr*-preserving conditional expectation.*

To prove this, we prove that E_N satisfies Definition 2.2.4 and the equation (2.2). In fact, E_N not only satisfies (2.2) but is uniquely characterised by it.

Proposition 2.2.7. *For all $x \in M$, $E_N(x)$ is the* unique *element of N such that*

$$\operatorname{tr}(E_N(x)y) = \operatorname{tr}(xy) \text{ for all } y \in N. \tag{2.3}$$

Proposition 2.2.7 turns out to imply Theorem 2.2.6, so we prove it first.

Proof. To show (2.3) holds, let $x \in M, y \in N$. Then,

$$\operatorname{tr}(E_N(x)y) = \langle \widehat{E_N(x)}, \widehat{y^*} \rangle = \langle e_N\hat{x}, \widehat{y^*} \rangle = \langle \hat{x}, e_N\widehat{y^*} = \langle \hat{x}, \widehat{y^*} \rangle = \operatorname{tr}(xy),$$

where the second equality is by Definition 2.2.3, the third as e_N is a projection, and the fourth as $\widehat{y^*} \in \hat{N} \subset L^2(N)$.

To prove uniqueness, suppose $z \in N$ satisfies $\operatorname{tr}(zy) = \operatorname{tr}(xy)$ for all $y \in N$. We need to show $z = E_N(x)$ or equivalently $\hat{z} = e_N\hat{x}$. To do this, it suffices to show $\hat{x} - \hat{z} \in L^2(N)^\perp$. Let $\hat{y} \in \hat{N}$ be arbitrary. Then,

$$\langle \hat{x} - \hat{z}, \hat{y} \rangle = \langle \hat{x}, \hat{y} \rangle - \langle \hat{z}, \hat{y} \rangle = \operatorname{tr}(xy^*) - \operatorname{tr}(zy^*) = \operatorname{tr}(xy^*) - \operatorname{tr}(xy^*) = 0.$$

As $\hat{N}$ is dense in $L^2(N)$, $\hat{x} - \hat{z} \in L^2(N)^\perp$, and we're done. $\square$

We can now sketch a proof of Theorem 2.2.6.

Proof. (Theorem 2.2.6)

We need only show that E_N is a conditional expectation, as Proposition 2.2.7 implies it is tr-preserving. All of the conditions of Definition 2.2.4 can be proven by leveraging the unique characterisation of E_N in Proposition 2.2.7, so we present only the proof of the bimodule property to illustrate the technique.

Let $x \in M$ and $a, b, y \in N$. Then,

$$\operatorname{tr}(aE_N(x)by) = \operatorname{tr}(E_N(x)bya) = \operatorname{tr}(xbya) = \operatorname{tr}((axb)y),$$

where the second equality is due to Proposition 2.2.7. This holds for all $y \in N$, and $aE_N(x)b \in N$, so $aE_N(x)b = E_N(axb)$ by the unique characterisation. □

In fact, by combining Theorem 2.2.6 and Proposition 2.2.7, we see that E_N is the *unique* tr-preserving conditional expectation $M \to N$. It enjoys several properties reminiscent of probability-theoretic expectations.

Proposition 2.2.8. *E_N acts as the identity on N, and $E_N : M \to N$ is a normal map. In addition, for all $x \in M$, and any tracial inclusion $P \subset N$, E_N satisfies the following.*

$$\operatorname{tr}(E_N(x)) = \operatorname{tr}(x). \tag{2.4}$$

$$E_N(x^*) = E_N(x)^*. \qquad \textit{(}*\textit{-preserving)} \tag{2.5}$$

$$E_N(x^*)E_N(x) \leq E_N(x^*x). \qquad \textit{(Jensen inequality)} \tag{2.6}$$

$$E_N(x^*x) \geq 0. \text{ Moreover, } E_N(x^*x) = 0 \implies x = 0. \qquad \textit{(Positive-definite)} \tag{2.7}$$

$$E_P(E_N(x)) = E_P(x). \qquad \textit{(Tower property)} \tag{2.8}$$

Proofs are found in [Kad04, p151]; see also [Ume54]. We omit them, as they are either simple applications of the unique characterisation of E_N (similar to the proof of Theorem 2.2.6), or adaptations of standard probability-theoretic proofs.

Armed with a thorough understanding of E_N, we can better understand the the Jones projection e_N and hence the basic construction $\langle M, e_N \rangle$.

Abstract properties of the basic construction

The following are the most important relations between e_N, N, and M.

Proposition 2.2.9. *[Jon83]*

1. *$e_N x e_N = E_N(x)e_N$ for all $x \in M$.*

2. If $x \in M$, then $x \in N \iff e_N x = x e_N$.

Proof. For part 1, it suffices to show both sides agree on the dense subspace $\hat{M}$ of $L^2(M)$. Let $\hat{y} \in \hat{M}$. Applying Definition 2.2.3 twice[1],

$$e_N x e_N \hat{y} = e_N x (E_N(y))\hat{} = (E_N(x E_N(y)))\hat{} = (E_N(x) E_N(y))\hat{}\,.$$

The last equality uses the bimodule property (Definition 2.2.4). But this is exactly equal to $E_N(x) e_N \hat{y}$ by Definition 2.2.3.

For the forward direction of part 2, suppose $x \in N$ and let $\hat{y} \in \hat{M}$. Then,

$$x e_N \hat{y} = (x E_N(y))\hat{} = (E_N(xy))\hat{} = e_N \widehat{xy} = (e_N x)\hat{y},$$

so e_N commutes with x. For the reverse direction, suppose $x \in M$ commutes with e_N. Then, $\hat{x} = x e_N \hat{1} = e_N x \hat{1} = e_N \hat{x}$, which implies $\hat{x} \in e_N \hat{M} = \hat{N}$, by Proposition 2.2.2. □

Corollary 2.2.10. *$e_N M e_N = N e_N$.*

Proof. E_N surjects onto N, so this is immediate from Proposition 2.2.9(1). □

For practical purposes, it is good to know an explicit dense $*$-subalgebra of $\langle M, e_N \rangle$.

Lemma 2.2.11. *$M + \operatorname{span} M e_N M = \operatorname{span}\{a + b e_N c : a, b, c \in M\}$ is a dense $*$-subalgebra of $\langle M, e_N \rangle$.*

Proof. By definition, the $*$-subalgebra $\langle M, e_N \rangle_{\text{alg}}$ generated by e_N and M is dense in $\langle M, e_N \rangle$. As $M \cup \{e_N\}$ is $*$-closed, $\langle M, e_N \rangle_{\text{alg}}$ is precisely the set of finite sums of finite-length words in e_N and M. As $e_N^2 = e_N$, such a word can be written to alternate between M and e_N.

By Corollary 2.2.10, we have that $e_N M e_N = N e_N \subset M e_N$. Using this fact, any word is reducible to the form $M, M e_N$, or $e_N M$. As $1 \in M$, the set of finite sums of such words is $M + \operatorname{span} M e_N M$. Hence $M + \operatorname{span} M e_N M$ is dense. □

We demonstrate the value of this explicit dense set in the following proof. As with any projection in a von Neumann algebra, it is important to know the cutdown of $\langle M, e_N \rangle$ by e_N. Since $e_N \in \langle M, e_N \rangle$ and $e_N \in N'$ (by Proposition 2.2.9), Proposition 1.3.7 implies that $N e_N$ and $e_N \langle M, e_N \rangle e_N$ are von Neumann algebras. In fact, they are identical.

[1] Note that we write $\hat{}$ as a superscript if it is applied to a very long expression.

Proposition 2.2.12. *$Ne_N = e_N\langle M, e_N\rangle e_N$ and hence $x \mapsto xe_N$ defines a von Neumann algebra isomorphism of N onto $e_N\langle M, e_N\rangle e_N$.*

Proof. Because $e_N \in N'$, Lemma 1.4.4 implies that $x \mapsto xe_N$ is an isomorphism of N onto its image. Hence it suffices to show the equality $Ne_N = e_N\langle M, e_N\rangle e_N$.

The inclusion $Ne_N = e_N N e_N \subset e_N\langle M, e_N\rangle e_N$ is obvious. To prove the reverse inclusion, it suffices to prove Ne_N contains a generating set for $e_N\langle M, e_N\rangle e_N$. By Lemma 2.2.11, such a generating set is $e_N\left(M + \operatorname{span} Me_N M\right)e_N$. Note this set can be rewritten:

$$\begin{aligned} e_N\left(M + \operatorname{span} Me_N M\right)e_N &= e_N M e_N + \operatorname{span} e_N M e_N M e_N \\ &= e_N M e_N + \operatorname{span}\left(e_N M e_N\right)^2 \\ &= Ne_N + \operatorname{span}(Ne_N)^2. \end{aligned}$$

The last equality holds by Corollary 2.2.10. As Ne_N is an algebra, $Ne_N + \operatorname{span}(Ne_N)^2 \subset Ne_N$, and so Ne_N contains a generating set for $e_N\langle M, e_N\rangle 1 e_N$. □

This shows that N and $\langle M, e_N\rangle$ cannot be distinguished by their action on the subspace $L^2(N)$. Their action on the full space $L^2(M)$ distinguishes them.

Corollary 2.2.13. *e_N is finite[2] in $\langle M, e_N\rangle$.*

Proof. All tracial von Neumann algebras are finite. To see this, note $1 \approx p$ implies $1 = uu^*$ and $p = u^*u$ for some partial isometry u, and therefore $\operatorname{tr}(1-p) = 0$, which implies $1 = p$ by positivity of the trace.

N is tracial, so 1 is finite in N. By applying the above isomorphism, we find that e_N is finite in $Ne_N = e_N\langle M, e_N\rangle e_N$. By definition, e_N is finite if and only if $e_N \not\approx p$ for any $p < e_N$. But any $p < e_N$ belongs to both $e_N\langle M, e_N\rangle e_N$ and $\langle M, e_N\rangle$, so it follows that e_N is finite in $\langle M, e_N\rangle$. □

These are most of the abstract algebraic properties necessary to probe the basic construction. However, as alluded to above, it is also essential to understand the spatial properties of its representation on $L^2(M)$.

Spatial properties of the basic construction

The Jones projection e_N provides a neat formula for N and N'. In this subsection, commutants are taken with respect to the action on $L^2(M)$.

[2] Recall Definition 1.3.5.

Lemma 2.2.14. *$N = (M' \cup \{e_N\})'$ and $N' = (M' \cup \{e_N\})''$.*

Proof. It is easy to see that the commutant of a union is an intersection of commutants. Then, $(M' \cup \{e_N\})' = M'' \cap \{e_N\}' = M \cap \{e_N\}'$. But $M \cap \{e_N\}' = N$, by Proposition 2.2.9(2), proving the first equality. The second is then immediate. □

These formulae are useful but arcane. We recast them into a more elegant form by recalling the antiunitary extension of the $*$-operation, $J : L^2(M) \to L^2(M)$ (Definition 1.6.6), and the injective antilinear morphism Ad J (Lemma 1.6.7).

Proposition 2.2.15. *$\langle M, e_N \rangle = JN'J$ and $\langle M, e_N \rangle' = JNJ$.*

In particular, $\langle M, e_N \rangle$ (resp. $\langle M, e_N \rangle'$) is spatially antilinearly isomorphic to N' (resp. N).

We can visualise this result. Recall (Proposition 1.6.9) that Ad J relates M to M' by $JMJ = M'$. Combined with the above, this means Ad J maps the triplet $N \subset M \subset \langle M, e_N \rangle$ to the triplet $\langle M, e_N \rangle' \subset M' \subset N'$. See below:

$$
\begin{array}{ccc}
N' & \xleftrightarrow{\operatorname{Ad} J} & \langle M, e_N \rangle \\
\uparrow & & \uparrow \\
M' & \xleftrightarrow{\operatorname{Ad} J} & M \\
\uparrow & & \uparrow \\
\langle M, e_N \rangle' & \xleftrightarrow{\operatorname{Ad} J} & N
\end{array}
$$

The vertical arrows are inclusions, and the horizontal arrows are antilinear isomorphisms. In fact, as they are induced by antiunitary conjugation, they are actually antilinear *spatial* isomorphisms. Owing to this diagram, we informally call the basic construction $\langle M, e_N \rangle$ the 'reflection of N across M', a perspective that provides much intuition.

Proof. (Proposition 2.2.15)

We first prove $\langle M, e_N \rangle = JN'J$. From Lemma 2.2.14, we have that $JN'J = J\,(M' \cup \{e_N\})''\,J$. It can be shown that, because J is antiunitary and $M' \cup \{e_N\}$ is a self-adjoint set, the bicommutant theorem implies we can pull Ad J through the double commutant:

$$JN'J = (JM'J \cup \{Je_NJ\})'' = (M \cup \{e_N\})'' = \langle M, e_N \rangle \tag{2.9}$$

where the third equality holds as $JM'J = M$ and e_N commutes with J because $L^2(N)$ is J-invariant.

It is also not too hard to show $\operatorname{Ad} J$ can be pulled through a commutant. Take the commutant of (2.9) and pull $\operatorname{Ad} J$ through; this proves $\langle M, e_N\rangle' = JNJ$. □

$\operatorname{Ad} J$ has a disadvantage in that it is defined spatially. However, it has an equivalent abstract expression when restricted to a centre. This result is not in [Jon83] but is in later works like [Pop90, p23].

Proposition 2.2.16. *The map* $\operatorname{Ad} J$ *is an antilinear isomorphism of* $Z(N)$ *onto* $Z(\langle M, e_N\rangle)$ *given by* $z \mapsto \tilde{z}$*, where* $\tilde{z}$ *in* $Z\left(\langle M, e_N\rangle\right)$ *is the unique element such that* $\tilde{z}e_N = z^*e_N$*.*

Proof. By Proposition 2.2.15,

$$JZ(N)J = J\left(N' \cap N\right)J = JN'J \cap JNJ = \langle M, e_N\rangle \cap \langle M, e_N\rangle' = Z(\langle M, e_N\rangle)$$

so $\operatorname{Ad} J$ is an antilinear isomorphism of $Z(N)$ onto $Z(\langle M, e_N\rangle)$.

Let $z \in Z(N)$. The map $\operatorname{Ad} J$ sends z to JzJ, so let $\tilde{z} := JzJ$. To justify the alternative characterisation, note that $\tilde{z}e_N = z^*e_N$ is equivalent to $\tilde{z}|_{L^2(N)} = z^*|_{L^2(N)}$. As $z \in Z(N)$, $z^* \in Z(N)$ as well, and hence left- and right-multiplication by z^* agree on $L^2(N)$. By Proposition 1.6.8, the right-multiplication operator associated to z^* is $JzJ = \tilde{z}$, so we have that $\tilde{z}|_{L^2(N)} = z^*|_{L^2(N)}$.

To show $\tilde{z}$ is unique for this identity, suppose $z' \in Z(\langle M, e_N\rangle)$ is such that $z'e_N = ze_N$. In particular, $z' \in \langle M, e_N\rangle' = JNJ$, so $z' = JwJ$ for some $w \in N$. In particular, $\widehat{w^*} = JwJ\hat{1} = z'\hat{1} = z'e_N\hat{1} = z^*e_N\hat{1} = \widehat{z^*}$, so $w = z$ and hence $z' = JwJ = JzJ = \tilde{z}$. □

This covers most of the properties of the triplet $N \subset M \subset \langle M, e_N\rangle$ which we need to apply routinely. Next, we will consider *iterating* the basic construction to create a tower of nested algebras.

2.3 Iterating the basic construction

This section introduces the *Jones tower*. Of all the subfactor-theoretic objects introduced in this thesis, the Jones tower is by far the most important.

Given a tracial inclusion $N \subset M$, we can perform the basic construction and build $\langle M, e_N\rangle$. *A priori*, $\langle M, e_N\rangle$ isn't tracial. However, if it can be equipped with a (positive faithful normal normalised) trace that extends the trace of M, then we can perform the basic construction on $M \subset \langle M, e_N\rangle$ and build $\langle\langle M, e_N\rangle, e_M\rangle$.

Under certain circumstances, we can iterate the basic construction *ad infinitum* to form a tower $N \subset M \subset \langle M, e_N \rangle \subset \langle \langle M, e_N \rangle, e_M \rangle \subset \ldots$ called the *Jones tower*. This technique originated in [Jon83].

Definition 2.3.1. Suppose $N \subset M$ is a tracial inclusion of von Neumann algebras. Let $M_{-1} = N, M_0 = M$, and $M_1 = \langle M, e_N \rangle$.

We say that a sequence of tracial von Neumann algebras $\{M_n\}_{n=-1}^{\infty}$, where $M_{n-1} \subset M_n$ is a trace-preserving inclusion for all $n \geq 0$, is a *pre-Jones tower* of $N \subset M$ if the following is satisfied for all $n \geq 1$:

$$M_{n+1} \text{ is the basic construction of } M_{n-1} \subset M_n.$$

That is, $M_{n+1} = \langle M_n, e_{M_{n-1}} \rangle$, where $e_{M_{n-1}}$ is the Jones projection of $L^2(M_n)$ onto $L^2(M_{n-1})$.

To simplify notation, we write $e_n := e_{M_{n-2}}$ for $n \geq 1$. This choice of indexing is convenient as it means $e_n \in M_n$.

We are interested in towers where the Jones projections have some compatibility with the trace.

Definition 2.3.2. Suppose $A \subset B \subset C$ is a triplet of tracial von Neumann algebras with a distinguished projection $e \in C$. It is said to have the Markov property with Markov modulus $\tau > 0$ if the following holds.

$$\mathrm{tr}_C(ex) = \tau \, \mathrm{tr}_B(x) \quad \text{for all } x \in B. \tag{2.10}$$

Definition 2.3.3. A pre-Jones tower $\{M_n\}_{n \geq -1}$ is said to be a *Jones tower* with Markov modulus τ if every triplet $M_{n-1} \subset M_n \subset M_{n+1}$, with distinguished projection e_{n+1}, has the Markov property with modulus τ.

We are really only interested in those tracial inclusions $N \subset M$ for which a unique[3] Jones tower exists. For such inclusions, the Jones tower is a canonical construction – in fact, a faithful construction, as $N \subset M$ can be recovered from the tower by discarding all but the lowest two levels. Therefore, instead of studying $N \subset M$, we can study its Jones tower, a far richer object.

For example, Jones [Jon83] proved that a unique tower exists when $N \subset M$ is a II_1 subfactor with $[M : N] < \infty$. This is essential to the proof of the Jones index theorem. A Jones tower contains a family of projections $\{e_n\}_{n \geq 1}$; we will

[3]Since the trace is part of the data of $N \subset M$, this means 'unique up to choice of trace.' However, when $N \subset M$ is a II_1 factor, the trace is unique.

show that the structure of this sequence forces the index to take one of the values specified in the index theorem (Theorem 2.1.1).

Hence, it is imperative to establish a general condition for a tracial inclusion $N \subset M$ to possess a unique Jones tower. We introduce a helpful notion:

Definition 2.3.4. Suppose $N \subset M$ is a tracial inclusion. Let $M_1 = \langle M, e_N \rangle$ be its basic construction. A positive faithful normal normalised trace $\overline{\mathrm{tr}} : M_1 \to \mathbb{C}$ is a τ-*Markov-extended* trace if

1. (Extension condition) $M \subset M_1$ is a trace-preserving inclusion.

2. (Markov condition) $N \subset M \subset^{e_N} M_1$ satisfies the Markov property with modulus τ.

If M_1 has a τ-Markov-extended trace tr_{M_1}, then, as $M \subset M_1$ is a tracial inclusion, we can perform the basic construction on $M \subset M_1$ to obtain M_2. If M_2 also has a τ-Markov-extended trace, then we can construct M_3, and so on. However, eventually some M_n may fail to have a τ-Markov-extended trace, and the procedure must terminate. Ideally, we'd be able to check if $N \subset M$ has a Jones tower without manually constructing any M_n.

We'd like to define a property, say $\mathcal{P}$, such that $N \subset M$ has a Jones tower whenever it satisfies $\mathcal{P}$. How can $\mathcal{P}$ control M_n by only controlling $N \subset M$? The solution is to define $\mathcal{P}$ so that it is *preserved under the basic construction*, and hence propagates up the tower. We thus introduce the notion of a *tower-building property.*

(There is no hope to define $\mathcal{P}$ that works for an *arbitrary* $N \subset M$, so we build in an assumption that $\mathcal{P}$ applies only to a subfamily[4] of tracial inclusions.)

Definition 2.3.5. Suppose $\mathcal{C}$ is a family of tracial inclusions. Suppose $\mathcal{T}$ is a nonnegative real-valued function on $\mathcal{C}$, where we write $\mathcal{T}_{N\subset M}$ to denote its value on $N \subset M$. Suppose $\mathcal{P}$ is a property that is true or false for any member of $\mathcal{C}$. We say $\mathcal{P}$ is $\mathcal{T}$-*tower-building* if, whenever $N \subset M$ belongs to $\mathcal{C}$ and has $\mathcal{P}$, the following hold:

1. (Markov-extension condition) There exists a $\mathcal{T}_{N\subset M}$-Markov-extended trace $\overline{\mathrm{tr}} : M_1 \to \mathbb{C}$.

2. (ion condition) $M \subset M_1$ belongs to $\mathcal{C}$, has $\mathcal{P}$, and $\mathcal{T}_{M\subset M_1} = \mathcal{T}_{N\subset N}$.

[4]For example, the family of all II_1 subfactors.

If $N \subset M$ satisfies $\mathcal{P}$, then $M \subset M_1$ is a tracial inclusion with $\mathcal{P}$, so we can construct M_2. Then $M_1 \subset M_2$ is itself a tracial inclusion with $\mathcal{P}$, so we can continue *indefinitely.* We obtain a Jones tower for $N \subset M$, with modulus $\mathcal{T}_{N\subset M}$.

This isn't enough for our purposes; we want to ensure that $N \subset M$ has a *unique* Jones tower. For that, we need to place additional assumptions on $\mathcal{P}$.

Definition 2.3.6. Suppose $\mathcal{C}, \mathcal{T}$ are as above, and suppose $\mathcal{P}$ is a $\mathcal{T}$-tower-building property. We say $\mathcal{P}$ is *recoverably* $\mathcal{T}$-tower-building if

$$N \subset M \text{ has } \mathcal{P} \iff \text{There exists a } \mathcal{T}_{N\subset M}\text{-Markov-extended trace } \overline{\mathrm{tr}} : M_1 \to \mathbb{C}.$$

We say $\mathcal{P}$ is *uniquely* $\mathcal{T}$-tower-building if

$$N \subset M \text{ has } \mathcal{P} \implies \text{There exists a \textit{unique} } \mathcal{T}_{N\subset M}\text{-Markov-extended trace } \overline{\mathrm{tr}} : M_1 \to \mathbb{C}.$$

We say $\mathcal{P}$ is a *good* $\mathcal{T}$-tower-building property if it is both recoverably and uniquely tower-building.

We claim that the presence of a good tower-building property guarantees the existence and uniqueness of the Jones tower. In fact, we can say more.

Theorem 2.3.7. *(Tower-building theorem)*

Suppose $\mathcal{C}$ is a family of tracial inclusions and $\mathcal{P}$ is a good $\mathcal{T}$-tower-building property on $\mathcal{C}$. Suppose $N \subset M$ belongs to $\mathcal{C}$. Then,

$$N \subset M \textit{ has } \mathcal{P} \iff N \textit{ has a unique Jones tower } \{M_n\}_{n\geq -1} \textit{ and its modulus is } \mathcal{T}_{N\subset M}.$$

This theorem generalises results found in [JS97, pp36-45] and [GHJ89, pp80-86].

Proof. As we have introduced the 'right' definitions, the proof is just a matter of unravelling them. We abbreviate $\mathcal{T}_{N\subset M}$ to τ.

In the discussion following Definition 2.3.5, we explained that, if $N \subset M$ has $\mathcal{P}$, then it has a Jones tower $\{M_n\}_{n\geq -1}$ of modulus τ. Note that we get an extra fact for free: the self-replication condition of Definition 2.3.5 implies that every inclusion $M_{n-1} \subset M_n$ belongs to $\mathcal{C}$ and has $\mathcal{P}$.

The remainder of the proof has two parts. In part 1, we prove the converse of the above: if $N \subset M$ has a Jones tower of modulus τ, then $N \subset M$ has $\mathcal{P}$. In part 2, we show that the tower $\{M_n\}_{n\geq -1}$ is unique.

Part 1: Suppose $N \subset M$ has a Jones tower $\{M_n\}_{n\geq -1}$ of modulus τ. By Definition 2.3.1 and Definition 2.3.3, this implies in particular that M_1 has a trace tr_{M_1}, $M \subset M_1$ is a trace-preserving inclusion, and $N \subset M \subset^{e_N} M_1$ has the Markov property with modulus τ. By Definition 2.3.4, this precisely means that tr_{M_1} is a τ-Markov-extended trace. Because $\mathcal{P}$ is recoverable, Definition 2.3.6 implies $N \subset M$ has $\mathcal{P}$.

Part 2: Suppose $N \subset M$ has $\mathcal{P}$. We already know it has a Jones tower $\{M_n\}$ of modulus τ from part 1. Now suppose $\{\tilde{M}_n\}_{n\geq -1}$ is another Jones tower for $N \subset M$, with *any* modulus. We will show that it is identical to $\{M_n\}$, i.e. the traces agree and the algebras agree.

Abbreviate the trace on M_n (respectively $\tilde{M}_n$) to tr_n (respectively $\widetilde{\mathrm{tr}}_n$). We induct on n. The (-1)st and 0th level of both towers are $N \subset M$, so $(\tilde{M}_i, \widetilde{\mathrm{tr}}_i) = (M_i, \mathrm{tr}_i)$ for $i = -1, 0$. For $n \geq 0$, assume $(\tilde{M}_m, \widetilde{\mathrm{tr}}_m) = (M_m, \mathrm{tr}_m)$ for $m \leq n$. That is, the two Jones towers agree up to level n and possibly diverge after that. We illustrate this in Figure 2.1, where hooked arrows are tracial inclusions.

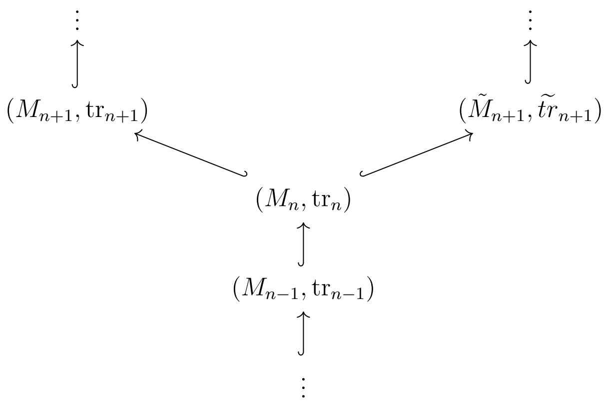


Figure 2.1: Jones towers diverging at level $n + 1$.

By part of the definition of a Jones tower (Definition 2.3.1), M_{n+1} and $\tilde{M}_{n+1}$ must both be the basic construction of $(M_{n-1} \subset M_n, \mathrm{tr}_n)$, so they're equal.

By another part of the definition (Definition 2.3.3), $M_{n-1} \subset M_n \subset^{e_{n+1}} \subset M_{n+1}$ has the Markov property with modulus τ. This must be true both when M_{n+1} is equipped with tr_{n+1} and when it is equipped with $\widetilde{\mathrm{tr}}_{n+1}$. By Definition 2.3.4, this means both traces are τ-Markov-extended.

However, at the start of the proof, we proved $M_{n-1} \subset M_n$ has $\mathcal{P}$. As $\mathcal{P}$ is uniquely tower-building, $\mathrm{tr}_{n+1} = \widetilde{\mathrm{tr}}_{n+1}$. By induction, $\{M_n\}$ is unique. □

Essentially, the tower-building theorem is the most general condition for existence and uniqueness of Jones towers. However, on its own, it can't be practically used to check if a given $N \subset M$ has a unique Jones tower. Rather, it is a 'meta-theorem' – a fundamental *recipe* for existence-uniqueness conditions. The theorem tells us that, if $\mathcal{C}$ admits a good tower-building property $\mathcal{P}$, then we can use $\mathcal{P}$ to find members of $\mathcal{C}$ with unique Jones towers.

To prove the Jones index theorem, we need to be able to build Jones towers for members of two families $\mathcal{C}$, for two quite different purposes. The first is of course the family of II_1 subfactors, and the second is the family of finite-dimensional tracial inclusions[5]. We will use the fundamental recipe to identify existence-uniqueness conditions for Jones towers in both families.

2.4 Jones tower of a II_1 subfactor

The results in this section were proven by Jones [Jon83, pp8-9].

Jones towers are immensely valuable to II_1 subfactor theory, as they unfold the structure of a subfactor $N \subset M$ to a far more expansive algebraic object.

Theorem 2.4.1. *Suppose $N \subset M$ is a II_1 subfactor. Then $[M : N] < \infty$ if and only if $N \subset M$ has a unique Jones tower $\{M_n\}_{n=-1}^{\infty}$.*

This theorem means that a finite-index subfactor and its Jones tower uniquely determine one another, but the tower contains far more visible data.

We prove this using the fundamental recipe. Let $\mathcal{C}$ be the collection of all II_1 subfactors. Let $\mathcal{T} : \mathcal{C} \to [0, \infty)$ evalute to $[M : N]^{-1}$ on $N \subset M$. Say $N \subset M$ has the 'finite index property' if $[M : N] < \infty$, and call this $\mathcal{P}$. If we can prove $\mathcal{P}$ is a good tower-building property, then applying the tower-building theorem (Theorem 2.3.7) allows us to conclude Theorem 2.4.1 immediately. (The conclusion occurs after Proposition 2.4.5.)

We first prove $\mathcal{P}$ is tower-building, then prove goodness afterwards. By Definition 2.3.5, we need to verify the Markov-extension condition and the self-replication condition. We first verify the latter, i.e. that $N \subset M$ having $\mathcal{P}$ implies $M \subset M_1$ belongs to $\mathcal{C}$, has $\mathcal{P}$, and has the same value of $\mathcal{T}$.

[5] $N \subset M$ where N, M are tracial and finite-dimensional.

Proposition 2.4.2. *If $N \subset M$ is a II_1 subfactor with finite index $[M : N] < \infty$, then $M \subset M_1$ is a II_1 subfactor with the same index.*

Proof. First, we prove that $M \subset M_1$ is a II_1 subfactor. It suffices to prove that M_1 is a II_1 factor, as inclusions of II_1 factors are automatically trace-preserving due to the uniqueness of the trace (Theorem 1.7.2).

From Proposition 2.2.16, M_1 and N have (antilinearly) isomorphic centres. As N is a factor, so is M_1. As $[M : N] < \infty$, N' is finite by Proposition 1.8.3. From Proposition 2.2.15, M_1 is (antilinearly) isomorphic to N', so it is itself finite. M_1 contains the infinite-dimensional subalgebra M, and so is itself infinite-dimensional. By the classification of factors, M_1 is a II_1 factor.

It remains to show $[M_1 : M] = [M : N]$. By Proposition 2.2.15, the inclusion $M \subset M_1$ is antilinearly isomorphic to $N' \subset M'$. Hence, $[M_1 : M] = [N' : M']$. By Proposition 1.8.3(5), $[N' : M'] = [M : N]$. □

This proves the self-replication condition. In particular, if $N \subset M$ has $\mathcal{P}$, then M_1 is equipped with a trace $\mathrm{tr}_{M_1} : M_1 \to \mathbb{C}$ by virtue of being a II_1 factor. To finish verifying that $\mathcal{P}$ is tower-building, we need to show that tr_{M_1} is Markov-extended (Definition 2.3.4). We already have that $M \subset M_1$ is trace-preserving, as all II_1 subfactors are. It remains to prove the Markov condition:

Proposition 2.4.3. *If $N \subset M$ has finite index, then $N \subset M \subset^{e_N} M_1$ is Markov with modulus $[M : N]^{-1}$.*

The proof of Proposition 2.4.3 relies on the following formula.

Proposition 2.4.4. *[Jon83, 3.1.7.]*
$\mathrm{tr}_{M_1}(e_N) = [M : N]^{-1}$.

Proof. From Proposition 1.7.8,

$$\dim_{e_N M_1 e_N}\left(e_N L^2(M)\right) = \mathrm{tr}_{M_1}(e_N)^{-1} \dim_{M_1} L^2(M) \tag{2.11}$$

First evaluate the left. Recall $x \mapsto x e_N$ is an isomorphism of N onto $e_N M_1 e_N$ by Proposition 2.2.12. As $e_N|_{L^2(N)}$ is the identity, this is a spatial isomorphism from the representation of N on $L^2(N)$ to $e_N M_1 e_N$ on $L^2(N)$. Hence,

$$\dim_{e_N M_1 e_N}\left(e_N L^2(M)\right) = \dim_N L^2(N) = 1. \tag{2.12}$$

To evaluate the right side of (2.11), we note

$$\dim_{M_1} L^2(M) = (\dim_{M_1'}(L^2(M)))^{-1} = (\dim_N L^2(M))^{-1} = [M : N]^{-1}. \tag{2.13}$$

The first equality holds by Proposition 1.7.8(3), and the second by Proposition 2.2.15, because Ad J is a spatial isomorphism and hence preserves coupling constants. The third is the definition of the index. Substituting (2.12) and (2.13) into (2.11) finishes the proof. □

This result is remarkable! By definition, the index is given as a coupling constant, which is *a priori* extremely difficult to compute. The simple formula above allows us to dispense with coupling constants entirely. Without this, classifying index values and proving the index theorem would be very technically challenging.

We obtain a proof of Proposition 2.4.3.

Proof. (Proposition 2.4.3)

We need to prove that $[M : N] < \infty$ implies following Markov relation:

$$\mathrm{tr}_{M_1}(e_N x) = [M : N]^{-1}\,\mathrm{tr}_M(x) \text{ for all } x \in M. \tag{2.14}$$

Define $\phi : N \to \mathbb{C}$ by $y \mapsto \mathrm{tr}_{M_1}(e_N y)$. ϕ is cyclic because $e_N \in N'$ (Proposition 2.2.9(2)). ϕ is normal because tr_{M_1} is. Hence ϕ is a normal, possibly non-normalised trace. By the uniqueness of the II_1 trace, ϕ must equal the trace tr_N up to a constant factor. Here we apply the remarkable formula obtained in Proposition 2.4.4. $\phi(1) = \mathrm{tr}_{M_1}(e_N) = [M : N]^{-1} = [M : N]^{-1}\,\mathrm{tr}_N(1)$, so this constant factor is $[M : N]^{-1}$. This proves (2.14) for $x \in N$.

To extend the proof of (2.14) to $x \in M$:

$$\mathrm{tr}_{M_1}(e_N x) = \mathrm{tr}_{M_1}(e_N^2 x) = \mathrm{tr}_{M_1}(e_N x e_N) = \mathrm{tr}_{M_1}(E_N(x) e_N)$$

by Proposition 2.2.9(1). As $E_N(x) \in N$, (2.14) applies.

$$\mathrm{tr}_{M_1}(e_N x) = [M : N]^{-1}\,\mathrm{tr}_N(E_N(x)) = [M : N]^{-1}\,\mathrm{tr}_M(x)$$

as E_N is trace-preserving. This proves (2.14) for all $x \in M$. □

Proposition 2.4.3 verifies that tr_{M_1} is a Markov-extended trace, which proves that $\mathcal{P}$ is tower-building. The final step is to show $\mathcal{P}$ is *good*.

Proposition 2.4.5. *$\mathcal{P}$ is a good $[M : N]^{-1}$-tower-building property.*

Proof. By Definition 2.3.6, we need to show $\mathcal{P}$ is uniquely and recoverably tower-building. M_1 is a II_1 factor, so it has a unique positive faithful normal normalised trace. Hence it's trivially true that $\mathcal{P}$ is uniquely tower-building.

We prove $\mathcal{P}$ is recoverable by contradiction. Suppose M_1 is tracial, with a $[M : N]^{-1}$-Markov-extended trace tr_{M_1}, but $N \subset M$ does not have $\mathcal{P}$. That is,

$[M : N] = \infty$. By Proposition 1.8.3(2), this means N' is infinite. But N' is antilinearly isomorphic to M_1 (Proposition 2.9). A tracial von Neumann algebra is finite[6], so we get a contradiction. □

Having identified a good tower-building property, the tower-building theorem (Theorem 2.3.7) implies Theorem 2.4.1. We hence obtain a unique Jones tower for every finite index subfactor $N \subset M$. Actually, we obtain a little more for free. Owing to Proposition 2.4.2, we have the following.

Theorem 2.4.6. *Suppose $N \subset M$ is a II_1 subfactor with $[M : N] < \infty$. Then $N \subset M$ has a unique Jones tower $\{M_n\}_{n \geq -1}$, and its modulus is $[M : N]^{-1}$.*

Moreover, for each $n \geq -1$, M_n is a II_1 factor and $[M_{n+1} : M_n] = [M : N]$.

The Jones tower is hence a natural method of making 'more of the same'.

Owing to the uniqueness of the Jones tower, we often think of a finite-index subfactor and its Jones tower as being the *same* object. In particular, the Jones tower reveals a simpler interpretation of the index – it encodes the Markov modulus of the tower, and no other data, and so is the 'simplest possible' invariant of a II_1 subfactor. Hence, restricting the allowable values of the index reduces to restricting the Markov modulus, which we do in Section 2.9.

2.5 Jones tower of a finite-dimensional inclusion

This section synthesises the original work of Jones [Jon83] with results that appear later, e.g. [JS97, pp36-45], [GHJ89, pp80-86].

We now consider the Jones towers of *finite-dimensional* tracial inclusions $A \subset B$. Recall that, when we write '$A \subset B$ is a tracial inclusion', there is implicitly a choice of trace tr_B on B, which is considered part of its data. Unlike the II_1 trace, this is a non-unique choice in general.

The *reason* we are interested in these Jones towers is quite different to the previous section. The point of building a Jones tower is not to understand $A \subset B$, as we already fully understand finite-dimensional inclusions. Rather, building towers *is* the point, as it's an easy way to build infinitely many objects from a finite-dimensional object. The main theorem of this section is the following:

[6] See proof of Proposition 2.2.13.

Theorem 2.5.1. *Suppose G is a nonempty connected bipartite graph. Then there exists a finite-dimensional tracial inclusion $A \subset B$ with Jones tower $\{B_n\}_{n=-1}^{\infty}$ of Markov modulus $\|G\|^{-2}$, where $\|G\|$ is the graph norm[7] of G.*

This means that we can tailor-make infinite towers of finite-dimensional algebras using only the data of a graph! Towards a proof, we need to express another existence-uniqueness condition for the Jones tower – this time, for finite-dimensional $A \subset B$ instead of subfactors $N \subset M$. Since the algebras in question are finite-dimensional, we can express such a condition with algebro-combinatorial objects. We use two kinds of objects: inclusion matrices, and *trace vectors*. We define the latter in Definition 2.5.4.

When we introduced the inclusion matrix, we had not yet introduced the basic construction, so we explain how they interact. Fix a finite-dimensional tracial inclusion $A \subset B$. If B_1 is its basic construction, what is the inclusion matrix of $B \subset B_1$?

Recall from Definition 1.5.8 that the row indices of $\Lambda_B^{B_1}$ are the minimal central projections (MCPs) of B, and the column indices are MCPs of B_1. Continuing the theme that B_1 'reflects' A, the MCPs of B_1 are determined by A.

Lemma 2.5.2. *Suppose P is the set of MCPs of A. Then $\tilde{P} := \{\tilde{p} : p \in P\}$ is the set of MCPs of B_1, where $\tilde{p} \in Z(B)$ is unique for the identity $pe_A = \tilde{p}e_A$.*

Proof. Immediate from Proposition 2.2.16. □

Then, for indexing purposes, $\tilde{P}$ (column indices of $\Lambda_B^{B_1}$) can be identified with P (row indices of Λ_A^B). Then the matrices $\Lambda_B^{B_1}$ and $\left(\Lambda_A^B\right)^T$ share row (and column) indices. Indeed, they are equal.

Lemma 2.5.3. $\Lambda_B^{B_1} = \left(\Lambda_A^B\right)^T$.

Proof. By Proposition 1.5.13, $\Lambda_{B'}^{A'} = (\Lambda_A^B)^T$. By Proposition 2.2.15, the inclusion $B' \subset A'$ is isomorphic to the inclusion $B \subset B_1$, so $\Lambda_{B'}^{A'} = \Lambda_B^{B_1}$. □

If $A \subset B$ is viewed as a 'plain' inclusion of finite-dimensional von Neumann algebras, it is completely described by the pair $(\Lambda_A^B, \vec{n}^A)$. However, we are viewing $A \subset B$ as a tracial inclusion, so we must also encode the trace data.

Let C be an arbitrary finite-dimensional von Neumann algebra, Q be its set of MCPs, and $\vec{n}^C \in \mathbb{N}^Q$ be its dimension vector. By Theorem 1.5.5, $C = \bigoplus_{q \in Q} Cq$, where $Cq \cong M_{n_q^C}(\mathbb{C})$. Observe the restriction of tr to Cq is also a trace, and

[7] Defined to be the norm of its adjacency matrix.

so must be proportional to the usual matrix trace Tr, where $\mathrm{Tr}(q) = n_q$ and $\mathrm{Tr}(r) = 1$ for any minimal projection $r \in Cq$. Therefore, tr is a linear combination of matrix traces on each simple summand of C.

$$\mathrm{tr}\left(\bigoplus_{q\in Q} x_q\right) := \sum_{q\in Q} \tau_q \,\mathrm{Tr}(x_q) \tag{2.15}$$

Hence any trace on C is uniquely determined by a vector.

Definition 2.5.4. If C is a finite-dimensional von Neumann algebra and Q is its set of MCPs, the *trace vector* of a trace $\mathrm{tr} : C \to \mathbb{C}$ is given by $\vec{\tau} = (\tau_p)_{p\in Q}$.

To distinguish traces on different algebras, write tr_C for a trace on C and $\vec{\tau}^C$ for its trace vector. We reduce trace properties to linear-algebraic identities.

Lemma 2.5.5. *(P.f.n. condition)*

$\mathrm{tr}_C : C \to \mathbb{C}$ *is a positive faithful normalised*[8] *trace if and only if* $\vec{\tau}^C$ *is a vector of positive entries such that* $\vec{\tau}^C \cdot \vec{n}^C = 1$.

Lemma 2.5.6. *(Extension condition)*

Suppose $C \subset D$ *are finite-dimensional von Neumann algebras with traces* tr_C *and* tr_D. *Then* $C \subset D$ *is a trace-preserving inclusion if and only if* $\Lambda_C^D \vec{\tau}^D = \vec{\tau}^C$.

Lemma 2.5.5 is easy to verify using (2.15). Lemma 2.5.6 is proven using the representation of $C \subset D$ from Theorem 1.5.7; it is not short, but is elementary.

These algebro-combinatorial objects – inclusion matrices and trace vectors – allow us to state what we claim is a tower-building property.

Definition 2.5.7. We say that a tracial inclusion of finite-dimensional von Neumann algebras $A \subset B$ is *Frobenius*[9] at parameter $\mu > 0$ if $\vec{\tau}^B$ is an eigenvector of $(\Lambda_A^B)^T \Lambda_A^B$ with eigenvalue μ.

We now state our existence-uniqueness condition.

Theorem 2.5.8. *Suppose* $A \subset B$ *is a tracial inclusion of finite-dimensional von Neumann algebras and* $\tau > 0$. *Then,*

$$A \subset B \text{ is Frobenius at } \tau^{-1} \iff A \subset B \text{ has a unique Jones tower and its modulus is } \tau.$$

[8]'Normal' can be omitted as it is redundant: any functional on a finite-dimensional von Neumann algebra is ultraweakly continuous.

[9]We choose this name as the Perron-Frobenius theorem (Theorem 2.5.10) provides examples of such eigenvectors.

Once again, we follow the fundamental recipe to prove this. Write $\mathcal{C}$ for the family of finite-dimensional tracial inclusions. Fix $\mathcal{T}$ to be a constant $\tau > 0$. Write $\mathcal{P}$ to mean the Frobenius property with τ^{-1}. By the tower-building theorem, proving Theorem 2.5.8 reduces to proving $\mathcal{P}$ is a good τ-tower-building property.

Proposition 2.5.9. *$\mathcal{P}$ is a good τ-tower-building property for $\mathcal{C}$.*

Individual parts of Theorem 2.5.8 are known in the literature, but not centralised. Existence was known to Jones [Jon83]. Uniqueness and/or exhaustiveness (i.e. that *all* $A \subset B$ with unique Jones tower are Frobenius) are found piecemeal in [Pop90, 2.3], [JS97, 3.2.5], etc., but proofs are missing or sparse. Hence we give a full proof of Proposition 2.5.9 (and hence Theorem 2.5.8).

Proof. We must show $\mathcal{P}$ is recoverably and uniquely tower-building. In fact, the 'unique' modifier will come for free, so we begin by proving $\mathcal{P}$ is recoverable.

By Definition 2.3.6, a recoverably tower-building property satisfies the self-replication condition and is *equivalent* to the Markov-extension condition. (These conditions are defined in Definition 2.3.5.) Therefore, we split the proof that $\mathcal{P}$ is recoverable into two parts.

Part 1: Our goal is to prove $\mathcal{P}$ is equivalent to the Markov-extension condition. By unfolding definitions, this is equivalent to:

$$\begin{aligned} A \subset B \text{ has } \mathcal{P} &\iff B_1 \text{ has a } \tau\text{-Markov-extended trace.} \\ &\iff B_1 \text{ has a p.f.n. trace such that } \mathrm{tr}_{B_1}|_B = \mathrm{tr}_B \\ &\qquad \text{and } A \subset B \overset{e_A}{\subset} B_1 \text{ is Markov with modulus } \tau \end{aligned} \tag{2.16}$$

where 'p.f.n.' means 'positive faithful normalised.' We can simplify this goal using a fact whose proof we defer to the end.

$$\vec{\tau}^{B_1} = \tau\vec{\tau}^A \iff A \subset B \overset{e_A}{\subset} B_1 \text{ is Markov with modulus } \tau. \tag{2.17}$$

Assuming this holds, (2.16) reduces to

$$\begin{aligned} A \subset B \text{ has } \mathcal{P} \iff\ & B_1 \text{ has a p.f.n. trace with trace vector } \vec{\tau}^{B_1} = \tau\vec{\tau}^A \\ & \text{and } \mathrm{tr}_{B_1}|_B = \mathrm{tr}_B. \end{aligned} \tag{2.18}$$

Therefore, the goal of Part 1 amounts to proving (2.18).

$\implies$: Define $\vec{\tau}^{B_1} := \tau\vec{\tau}^A$. It suffices to verify that the p.f.n. condition (Lemma 2.5.5) and extension condition (Lemma 2.5.6) hold. To verify the extension condition, note

$$\Lambda_B^{B_1}\vec{\tau}^{B_1} = (\Lambda_A^B)^T\vec{\tau}^{B_1} = \tau(\Lambda_A^B)^T\vec{\tau}^A$$

by Lemma 2.5.3. As $A \subset B$ is a trace-preserving inclusion by assumption, it satisfies the extension condition, so $\Lambda_A^B \vec{\tau}^B = \vec{\tau}^A$. Hence,

$$\Lambda_B^{B_1} \vec{\tau}^{B_1} = \tau (\Lambda_A^B)^T \Lambda_A^B \vec{\tau}^B = \vec{\tau}^B \tag{2.19}$$

This verifies the extension condition on $B \subset B_1$.

It remains to verify the p.f.n. condition. A is tracial by assumption – by definition this means its trace tr_A is p.f.n. By the p.f.n. condition $\vec{\tau}^A$ has positive entries, and so the same is true of $\vec{\tau}^{B_1} = \tau\vec{\tau}^A$. Moreover,

$$\vec{n}^{B_1} \cdot \vec{\tau}^{B_1} = (\vec{n}^A \Lambda_A^B \Lambda_B^{B_1}) \cdot \vec{\tau}^{B_1} = \vec{n}^A \cdot (\Lambda_A^B \Lambda_B^{B_1} \vec{\tau}^{B_1}) = \vec{n}^A \cdot \vec{\tau}^A = 1.$$

The first equality comes from applying Lemma 1.5.11 twice. The third equality is an application of (2.19) and the identity $\vec{\tau}^{B_1} = \tau\vec{\tau}^A$. The final equality is due to the p.f.n. condition. Hence, $\vec{n}^{B_1} \cdot \vec{\tau}^{B_1} = 1$, and $\vec{n}^{B_1}$ has positive entries, so (B_1, tr_{B_1}) satisfies the p.f.n. condition. This proves one direction of (2.18).

$\Longleftarrow$: Assume B_1 has a p.f.n. trace with $\vec{\tau}^{B_1} = \tau\vec{\tau}^A$ and $\mathrm{tr}_{B_1}|_B = \mathrm{tr}_B$. The latter identity implies $B \subset B_1$ satisfies the extension condition. Then,

$$\begin{aligned} & \vec{\tau}^{B_1} = \tau\vec{\tau}^A && \\ \Longrightarrow\ & \vec{\tau}^{B_1} = \tau \Lambda_A^B \vec{\tau}^B && \text{(Extension condition on } A \subset B) \\ \Longrightarrow\ & \Lambda_B^{B_1} \vec{\tau}^{B_1} = \tau (\Lambda_A^B)^T \Lambda_A^B \vec{\tau}^B && \text{(Lemma 2.5.3)} \\ \Longrightarrow\ & \tau^{-1} \vec{\tau}^B = (\Lambda_A^B)^T \Lambda_A^B \vec{\tau}^B && \text{(Extension condition on } B \subset B_1). \end{aligned}$$

The last equality is precisely what it means for $A \subset B$ to be Frobenius with parameter τ^{-1}, i.e. to have $\mathcal{P}$. This proves (2.18). To finish Part 1 of this proof, it remains to prove the fact whose proof we deferred, (2.17).

The Markov relation is $\mathrm{tr}_{B_1}(xe_A) = \tau\,\mathrm{tr}_B(x)$ for all $x \in B$. By Proposition 2.2.9, $\mathrm{tr}_{B_1}(xe_A) = \mathrm{tr}_{B_1}(e_A x e_A) = \mathrm{tr}_{B_1}(E_A(x)e_A)$. As $\operatorname{im} E_A = A$, it follows that

$$\mathrm{tr}_{B_1}(xe_A) = \tau\,\mathrm{tr}_A(x)\ \forall\ x \in B \iff \mathrm{tr}_{B_1}(xe_A) = \tau\,\mathrm{tr}_A(x)\ \forall\ x \in A.$$

Note that a trace is determined by the values it takes on minimal projections[10] (this is obvious in $M_n(\mathbb{C})$, for example). As $x \mapsto xe_A$ is an isomorphism of A onto $e_A B_1 e_A$ (Proposition 2.2.12), it preserves minimal projections. Then the minimal projections of $e_A B_1 e_A$ are $\{re_A : r \text{ is minimal in } A.\}$. It follows that

$$\begin{aligned} \mathrm{tr}_{B_1}(xe_A) = \tau\,\mathrm{tr}_A(x)\ \forall\ x \in B &\iff \mathrm{tr}_{B_1}(re_A) = \tau\,\mathrm{tr}_A(r) \text{ for minimal } r \in A. \\ &\iff \vec{\tau}^{B_1} = \tau\vec{\tau}^A \end{aligned}$$

[10]Not to be confused with minimal central projections. E.g. $M_3(\mathbb{C})$ has one minimal central projection (the identity) and many minimal (rank-one) projections.

where the last equivalence can be seen by inspecting Definition 2.5.4 and the remarks preceding it. This proves (2.17), as required, concluding Part 1.

Part 2: The goal of Part 2 is to prove $\mathcal{P}$ satisfies the self-replication condition (as defined in Definition 2.3.5). Suppose $A \subset B$ has $\mathcal{P}$.

It is clear that B_1 is finite-dimensional[11]. Then $B \subset B_1$ is automatically in $\mathcal{C}$ (the family of finite-dimensional tracial inclusions).

The nontrivial part is showing $B \subset B_1$ also has $\mathcal{P}$. We have already proven that, if $A \subset B$ has $\mathcal{P}$, then $\mathrm{tr}_{B_1}|_B = \mathrm{tr}_B$ and $\vec{\tau}^{B_1} = \tau\vec{\tau}^A$ (see (2.16) and (2.17)). It follows that

$$(\Lambda_B^{B_1})^T \Lambda_B^{B_1} \vec{\tau}^{B_1} = \Lambda_A^B \Lambda_B^{B_1} \vec{\tau}^{B_1} = \Lambda_A^B \vec{\tau}^B = \vec{\tau}^A = \tau^{-1} \vec{\tau}^{B_1}.$$

where the second and third equalities are two applications of the extension condition. This equality is precisely what it means for $B \subset B_1$ to have $\mathcal{P}$, by Definition 2.5.7. This concludes Part 2; combined with Part 1, it implies $\mathcal{P}$ is recoverably τ-tower-building.

By (2.17), $\vec{\tau}^{B_1} := \tau\vec{\tau}^A$ is the only choice of trace vector for B_1 such that $A \subset B \subset^{e_A} B_1$ is Markov with modulus τ. Hence, by definition, $\mathcal{P}$ is also uniquely τ-tower-building. We conclude $\mathcal{P}$ is a good τ-tower-building property. □

Applying the tower-building theorem (Theorem 2.3.7) then yields a proof of Theorem 2.5.8. We hence have an existence-uniqueness condition for Jones towers of finite-dimensional inclusions. That is, whenever $A \subset B$ is Frobenius, we automatically obtain a unique Jones tower of finite-dimensional algebras. This condition is especially powerful because the Frobenius property is purely linear-algebraic. We apply a powerful result from linear algebra, the *Perron-Frobenius theorem*, to construct Jones towers with ease.

Theorem 2.5.10. *(Perron-Frobenius)*

Suppose Ξ is the adjacency matrix of a nonempty connected graph. Then Ξ has a unique (up to positive scalar multiplication) eigenvector $\vec{\tau}$ with positive entries. The eigenvalue of $\vec{\tau}$ is $\|\Xi\|$.

See [Gan59, p53],[GHJ89, pp13-14]. We now prove Theorem 2.5.1, restated here:

Theorem 2.5.11. *Suppose G is a nonempty connected bipartite graph. Then there exists a finite-dimensional tracial inclusion $A \subset B$ such that its Jones tower $\{B_n\}_{n=-1}^{\infty}$ has Markov modulus $\|G\|^{-2}$.*

[11]Because it is a subalgebra of $L^2(B)$, which is finite-dimensional due to the finite-dimensionality of B.

Proof. Suppose G has left vertices P and right vertices Q, where $|P| = n$, $|Q| = m$. Let Λ be the $(n \times m)$ biadjacency matrix of G. Let $A = \mathbb{C}^{\oplus n}$, so it has dimension vector $\vec{n}^A = \underbrace{(1, \dots, 1)}_{n \text{ entries}}$, and let B be defined by the dimension vector $\vec{n}^B = \vec{n}^A \Lambda$. This is consistent with the constraints of Lemma 1.5.11, and it is easy to construct an inclusion map $\iota : A \to B$ such that the Bratteli diagram of $A \subset B$ is G, and hence their inclusion matrix is Λ.

To apply the existence-uniqueness condition and show $A \subset B$ has a Jones tower, we will equip B with a p.f.n. trace, then show that $A \subset B$ is Frobenius.

Define a graph H on Q by drawing an edge between each pair $q, q' \in Q$ for each path of length 2 between q and q' in G. Then the *adjacency* matrix of H is $\Lambda^T \Lambda$. It's clear H is connected whenever G is. Hence, by Theorem 2.5.10, $\Lambda^T \Lambda$ has a Perron-Frobenius eigenvector $\vec{\tau}$ with eigenvalue $\|H\| = \|\Lambda\|^2 = \|G\|^2$.

Let $\vec{\tau}^B = \vec{\tau}$ and assume it is scaled so that $\vec{n}^B \cdot \vec{\tau}^B = 1$. As $\vec{\tau}^B$ has positive entries, it then follows from the p.f.n. condition (Lemma 2.5.5) that it defines a p.f.n. trace tr_B. This equips $A \subset B$ with the structure of a tracial inclusion. As $\vec{\tau}^B$ is an eigenvector of Λ with positive eigenvalue $\|G\|^2$, it follows that $A \subset B$ is Frobenius at $\|G\|^2$. By Theorem 2.5.8, we conclude $A \subset B$ has a Jones tower with modulus $\|G\|^{-2}$. $\square$

Consider an example: the linear graph on $n - 1$ vertices, also known as the Coxeter diagram A_{n-1}, satisfies $\|A_{n-1}\|^2 = 4\cos^2(\pi/n)$ [GHJ89, 1.4.3]. The cases where $n = 1, 2$ are degenerate, but when $n \geq 3$, Theorem 2.5.11 shows that we can build a Jones tower with modulus $\tau = 1/\left(4\cos^2(\pi/n)\right)$ for $n \geq 3$.

We will see in Section 2.8 that a Jones tower with Markov modulus τ contains a II_1 subfactor with index τ^{-1}. Hence, the A_n Coxeter diagrams are responsible for the subfactors with index in the discrete series of Theorem 2.1.1!

2.6 Relations in the Jones tower

This concludes our study of the existence and uniqueness theory of Jones towers. In particular, we've shown that 'almost all' II_1 subfactors have unique Jones towers; only infinite-index subfactors do not. This is why the Jones tower is integral to all modern II_1 subfactor theory; it unpacks extra structure from a subfactor, without costing extra data.

In particular, it is invaluable for proving the index theorem. In anticipation of the proof, we introduce key algebraic properties of the Jones tower. The chief property is that the sequence $\{e_1, e_2, \ldots\}$ satisfies *Jones relations.* Much of the Jones tower's algebraic structure is owed to these relations – i.e., many of its properties are enjoyed by any family of projections with the same relations. Hence, all we do in Section 2.6 is identify the relations. In Section 2.7, we abstract away from the Jones tower and study the relations in a vacuum.

Remark 2.6.1. We often write $N \subset M \subset M_1 \subset M_2 \subset \ldots$ to describe the Jones tower. Indeed, one can build a tracial von Neumann algebra M_∞ that contains M_n for all $n \geq -1$. This is done using a more general case of the GNS construction than the one we presented in Section 1.6. One uses the construction to represent $\bigcup_{n=-1}^{\infty} M_n$ on a Hilbert space, then defines M_∞ to be its weak closure. Hence we view the entire Jones tower as living in M_∞.

Fix a tracial inclusion $N \subset M$. Assume it has a unique Jones tower $\{M_n\}_{n=-1}^{\infty}$, and its Markov modulus is τ. Recall, by Definition 2.3.1, $M_{n-2} \subset M_{n-1} \subset^{e_n} M_n$ is a basic construction triplet for all $n \geq 1$. Hence all results from Section 2.2 apply by replacing $N, M, \langle M, e_N \rangle$ with M_{n-2}, M_{n-1}, M_n, and e_N with e_n[12]. First, we need a formula for conditional expectations in the tower.

Proposition 2.6.2. *For all* $x, y \in M_{n-1}$, $E_{M_{n-1}}(xe_ny) = \tau xy$.

Proof. By Proposition 2.2.7, $E_{M_{n-1}}(xe_ny)$ is the unique element of M_{n-1} satisfying $\mathrm{tr}_{M_n}((xe_ny)z) = \mathrm{tr}_{M_{n-1}}\big(E_{M_{n-1}}(xe_ny)z\big)$ for all $z \in M_{n-1}$. We compute:

$$\mathrm{tr}_{M_n}((xe_ny)z) = \mathrm{tr}_{M_n}(e_nyzx) = \tau\, \mathrm{tr}_{M_{n-1}}(yzx) = \mathrm{tr}_{M_{n-1}}(\tau xyz)$$

where the second equality is due to the Markov property of $M_{n-2} \subset M_{n-1} \subset^{e_n} M_n$ (Definition 2.3.2). This holds for all $z \in M_{n-1}$, proving $E_{M_{n-1}}(xe_ny) = \tau xy$. □

Corollary 2.6.3. $E_{M_{n-1}}(e_n) = \tau$.

This is enough to identify the Jones relations.

Theorem 2.6.4. *[Jon83, 3.4.2,3.1.7]*

For all $n \geq 1$, the following relations hold[13] *in* M_∞.

1. $e_n^2 = 1$.

2. $e_ne_{n+1}e_n = \tau e_n$ *and* $e_{n+1}e_ne_{n+1} = \tau e_{n+1}$.

[12] Recall e_n is just notation for the Jones projection $e_{M_{n-2}} : L^2(M_{n-1}) \to L^2(M_{n-2})$.

[13] As there is no Jones projection e_0, relation 2 is understood to hold only if it makes sense.

3. $e_n e_m = e_m e_n$ *if* $|n-m| > 1$.

Proof. Relation 1 is immediate as e_n is a projection. For relation 2, we compute

$$e_{n+1}e_ne_{n+1} = E_{M_{n-1}}(e_n)e_{n+1} = \tau e_{n+1}$$

where the first equality is by Proposition 2.2.9(1) and the second by Corollary 2.6.3. The second part of relation 2 is slightly more involved, but similar [Jon83, 3.4.1(ii)]. For relation 3, suppose $n < m$. Then $n \leq m-2$, so $e_n \in M_n \subset M_{m-2}$. Proposition 2.2.9(2) states that elements of M_{m-2} commute with e_m. □

Most of the structure of Jones towers stems directly from these relations, so we abstract away from the tower, and study the relations directly.

2.7 The Jones relations

We model a definition on the result of Theorem 2.6.4.

Definition 2.7.1. A family of projections $\{\varepsilon_i\}_{i=1}^{\infty}$ in a tracial von Neumann algebra $\mathcal{M}$ is said to satisfy the *Jones relations* with Markov modulus τ if it satisfies the following for $i, j \geq 1$:

1. $\varepsilon_i^2 = 1$,

2. $\varepsilon_i\varepsilon_{i\pm1}\varepsilon_i = \tau\varepsilon_i$,

3. $\varepsilon_i\varepsilon_j = \varepsilon_i\varepsilon_j$ if $|i-j| > 1$,

in addition to the Markov property:

4. $\mathrm{tr}(x\varepsilon_i) = \tau\,\mathrm{tr}(x)$ if $x \in [1 : i-1]$ (notation defined below).

We write $[n : m] = \langle 1, \varepsilon_n, \ldots, \varepsilon_m\rangle_{\mathrm{alg}}$ for the $*$-subalgebra of $\mathcal{M}$ generated by $1, \varepsilon_n, \ldots, \varepsilon_m$, where $n \leq m \in \mathbb{N} \cup \{\infty\}$. If we need to refer to relations 1-3 separately from relation 4, we call them the *algebraic Jones relations.*

Remark 2.7.2. (The interval notation)

1. If $n > m$, let $[n : m]$ denote the trivial von Neumann algebra $\mathbb{C}1$.

2. By the third Jones relation, $[n : m]$ and $[k : l]$ mutually commute if $k \geq m+2$.

Retroactively, Theorem 2.6.4 is re-stated as follows:

Theorem 2.7.3. *If* $\{M_n\}_{n=-1}^{\infty}$ *is a Jones tower of modulus* τ*, then the family* $\{e_i\}_{i=1}^{\infty} \subset M_\infty$ *satisfies the Jones relations with modulus* τ*.*

In particular, by Theorem 2.4.6, if $\{M_n\}$ is the unique Jones tower of a finite index subfactor, then $\{e_i\}_{i=1}^{\infty}$ satisfies the Jones relations with modulus $[M : N]^{-1}$.

However, these relations enjoy a wealth of inherent structure, whether they appear in a Jones tower or not. Their study is called the theory of Temperley-Lieb algebras, after Neville Temperley and Elliott Lieb, who independently studied the relations for their applications to statistical mechanics [TL71].

In this section, we fix arbitrary $\mathcal{M}$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ satisfying the Jones relations with modulus τ. We stress the difference between ε_i and the e_i. By definition, e_i is an element of the ith level (M_i) of some Jones tower; it has an explicit definition as some projection. Theorem 2.7.3 means that they satisfy the Jones relations. In contrast, we *assume* that the ε_i satisfy the Jones relations and there are no other constraints placed on them. The point is to abstractly study the Jones relations without extraneous structure.

This leap of abstraction provides a better perspective for proving the Jones index theorem. To prove that the index of a subfactor must take a value in $\{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$, which is a *subfactor-theoretic* fact, we will prove that $\tau^{-1} \in \{4\cos^2(\pi/n) : n \ldots\} \cup [4, \infty)$, which is a fact about the Jones relations. Conversely, to construct a subfactor with index having every value in $\{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$, we will build a subfactor out of $\{\varepsilon_i\}_{i=1}^{\infty}$.

Hence, we present a lengthy study of the Jones relations, as a prerequisite to the proof of the index theorem. Our writing in Section 2.7 synthesises [Jon83] with theory from [Wen87], [Kau90], and [Pen14].

The Temperley-Lieb algebra

To study the Jones relations, we define the universal object on the relations. (Recall from standard algebra that 'universal' refers, loosely, to the most general object for the given set of generators and relations.)

Definition 2.7.4. $TL(\tau)$, the *Temperley-Lieb algebra on n generators and parameter* τ is the universal $*$-algebra with identity 1 and generators $\{E_n\}_{n=1}^{\infty}$ satisfying $E_i^* = E_i$ in addition to the *Temperley-Lieb relations*:

1. $E_i^2 = \tau^{-1/2} E_i$.

2. $E_iE_{i\pm1}E_i = E_i$.

3. $E_iE_j = E_jE_i$ if $|i-j| > 1$.

Let $TL_{n:m}(\tau) := \langle 1, E_n, \ldots, E_m\rangle_{\text{alg}}$ and abbreviate $TL_{1:m}(\tau)$ to $TL_m(\tau)$.

We will typically write TL in place of $TL(\tau)$. Note that the Temperley-Lieb relations are related to the algebraic Jones relations by a scaling: it is easy to check that the family $\{\tau^{-1/2}e_i\}_{i=1}^{\infty}$ satisfies the Temperley-Lieb relations. Therefore, by universality, this family generates a representation[14] of TL.

Proposition 2.7.5. *There exists a unique surjective unital $*$-homomorphism $\varphi : TL(\tau) \to [1:\infty]$ that maps $E_i \mapsto \tau^{-1/2}\varepsilon_i$.*

The advantage of this representation is not immediately clear. For instance, $TL(\tau)$ lacks the analytical structure of $\mathcal{M}$, e.g. its topology and trace.

The true advantage is the remarkable fact that TL is isomorphic to an algebra of *diagrams*. We define a *graphical Temperley-Lieb algebra* as follows. As a vector space, define $TL_n^{\text{gr}}(\tau)$ as the free $\mathbb{C}$-vector space generated by planar isotopy classes of rectangular diagrams such as the following:

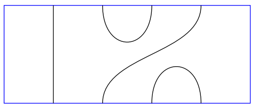


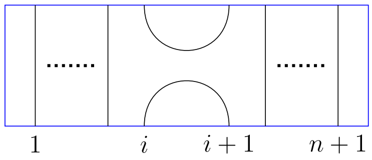
Figure 2.2: A diagram in TL_3^{gr}.

Each diagram has $n+1$ mobile strands with fixed endpoints on a fixed rectangle. (We omit or include the blue rectangle as necessary for illustration.) Diagrams may contain unfixed loops, but a loop is given a scalar value of $\tau^{-1/2}$. (A diagram with a loop is $\tau^{-1/2}$ times the same diagram with a loop removed.) Consider diagrams identical if they're isotopic by pulling strands without crossing.

If U, V are diagrams, define UV as the diagram formed by stacking U on top of V. The identity element is the diagram with $n+1$ vertical strands. If U is a diagram, then U^* is its reflection across a horizontal axis. The conjugate-linear extension of $*$ to $TL_n^{\text{gr}}(\tau) \to TL_n^{\text{gr}}(\tau)$ is a $*$-operation. This makes $TL_n^{\text{gr}}(\tau)$ a unital $*$-algebra called the *graphical Temperley-Lieb algebra on n generators*. The generators in question are the following:

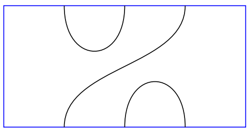
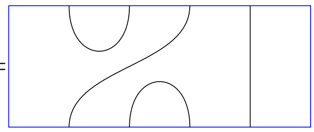
Definition 2.7.6. For $i = 1, \ldots, n$, define $E_i^{\text{gr}} \in TL_n^{\text{gr}}(\tau)$ as the following diagram on $(n+1)$ strands:

[14] Here we are using 'representation' to mean a representation of unital $*$-algebras, i.e. a unital $*$-homomorphism.

$E_i^{\text{gr}} :=$ 

We omit the proof that these generate TL_n^{gr}; see [Kau90].

Define an inclusion map from $(n+1)$-strand diagrams to $n+2$-strand diagrams by adding a strand to the right.

$U =$  $\iota_2(U) =$ 

This extends linearly to a $*$-homomorphism $\iota_n : TL_n(\tau) \to TL_{n+1}(\tau)$.

Let $TL^{\text{gr}}(\tau) := \bigcup_{n=0}^{\infty} TL_n(\tau)$, modulo the identification of each $U \in TL_n^{\text{gr}}$ with $\iota_n(U) \in TL_{n+1}^{\text{gr}}$. In fact, it is isomorphic to TL.

Theorem 2.7.7. *The unital $*$-homomorphism defined by $E_i \mapsto E_i^{\text{gr}}$ is a $*$-isomorphism of $TL(\tau)$ onto $TL^{\text{gr}}(\tau)$.*

Proof. We give a sketch. By universality of $TL(\tau)$, the map exists if the E_i^{gr} satisfy the Temperley-Lieb relations (as defined in Definition 2.7.4). As the $*$-operation is reflection about a horizontal axis, it's clear that the E_i^{gr} are self-adjoint. The third relation is easy to prove. For the first two, see Figure 2.3.

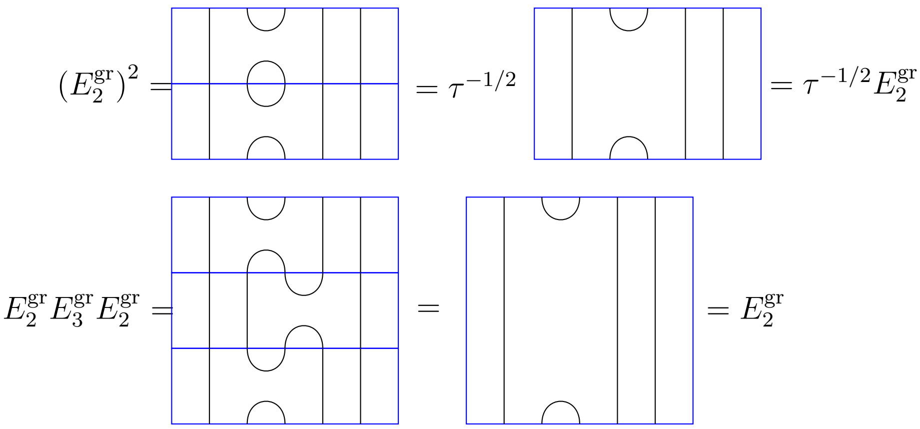


Figure 2.3: Visual demonstration that the E_i^{gr} satisfy Temperley-Lieb relations.

The bijectivity of the map is nontrivial; see [Jon14, pp30-32], [Kau90]. □

This is remarkable, and non-obvious. We can hence identify TL^{gr} with TL, and work with Temperley-Lieb diagrams instead of abstract algebraic elements.

In retrospect, this makes Proposition 2.7.5 an extremely potent result. The existence of the representation $\varphi : TL \to [1 : \infty]$ means that $[1 : \infty]$ is a representation of an algebra of diagrams, and we can reason about $\{\varepsilon_i\}_{i=1}^{\infty}$ and the Jones relations *graphically*, instead of algebraically! This technique was unavailable to Jones at the time of [Jon83]; the isomorphism $TL \cong TL^{\mathrm{gr}}$ is due to Louis Kauffman [Kau90]. We will use graphical techniques to 'update' the tools Jones used to prove the index theorem.

Before we study them further, we summarise the algebraic objects we have introduced. $[1 : \infty]$ is the unital $*$-subalgebra generated by $\{\varepsilon_i\}_{i=1}^{\infty}$ in some tracial von Neumann algebra $\mathcal{M}$, satisfying the Jones relations. $[n : m]$ denotes the $*$-subalgebra generated by $1, \varepsilon_n, \dots, \varepsilon_m$.

$TL(\tau)$ is the *universal* unital $*$-algebra on generators $\{E_i\}_{i=1}^{\infty}$ satisfying the Temperley-Lieb relations, which are scaled versions of the algebraic Jones relations. $TL_{n:m}(\tau)$ denotes the $*$-subalgebra generated by $1, E_n, \dots, E_m$.

The objects are related by the representation $\varphi : TL(\tau) \to [1 : \infty]$. Unlike the E_i, the ε_i are assumed to live in a tracial von Neumann algebra. Also, the ε_i may satisfy relations which do not follow from the Jones relations, whereas all relations satisfied by the E_i follow from Temperley-Lieb.

We will introduce graphical tools which help us study $\{\varepsilon_i\}_{i=1}^{\infty}$, and eventually prove the index theorem. However, before that, we need some algebraic results.

Algebraic relations

Suppose W, W' (w, w') are arbitrary words in the symbols E_i (ε_i). We write $W \sim W'$ $(w \sim w')$ if they are equal under the Temperley-Lieb (algebraic Jones) relations, up to scalar multiplication. We say that a word W is *reduced* if it has minimal length in its equivalence class.

Proposition 2.7.8. *Let n, m be finite. If W is a reduced word in $E_n, \dots, E_m$, then E_n, E_m each appear at most once in W. The same result holds if the E_i are exchanged for ε_i.*

Proof. Assume without loss of generality that $n = 1$. The Temperley-Lieb relations (Definition 2.7.4) are symmetric under reversing the order of indices, so we need only show there is at most one copy of E_m.

$E_1^2 = E_1$ implies the result when $m = 1$. Induct on m. We'll prove the contrapositive, so suppose W has more than one copy of E_m. By passing to a subword, assume $W = E_m U E_m$, where U is a word in $E_1, \dots, E_{m-1}$. We assume U is reduced; by induction, U contains one or zero copies of E_{m-1}. If zero, then (by the third Temperley-Lieb relation) E_m commutes with U and so $W \sim E_m^2 U \sim E_m U$, and hence W is unreduced. If one, $W = E_m U_1 E_{m-1} U_2 E_m$, where U_1, U_2 are words in $E_1, \ldots, E_{m-2}$. As E_m commutes with U_1 and U_2, we have that $W \sim U_1 E_m E_{m-1} E_m U_2 \sim U_1 E_m U_2$, proving W is unreduced. □

Corollary 2.7.9. $[n:m] = \langle 1, \varepsilon_n, \varepsilon_{n+1}, \ldots, \varepsilon_m \rangle_{\text{alg}}$ *is finite-dimensional if* n, m *are finite. Consequently,* $[n:m] = \{\varepsilon_n, \ldots, \varepsilon_m\}''$.

Proof. An induction argument using Proposition 2.7.8 shows that reduced words in $\varepsilon_n, \ldots, \varepsilon_m$ have a maximum length, proving $[n:m]$ is finite-dimensional. Finite-dimensional subspaces are weakly closed, so the other result follows. □

This result is important, as it means that the $[n:m]$ are finite-dimensional von Neumann algebras. In fact, they're tracial, as they inherit a trace from $\mathcal{M}$. We are interested in the $[n:m]$ as we will use them to construct II_1 factors by taking $m \to \infty$. Hence, we will establish their tracial and algebraic properties.

First note that, if $\{M_n\}$ is a Jones tower, $\mathcal{M} = M_\infty$, and $\varepsilon_i = e_i$, then $[n:m] \subset M_m$, by Definition 2.3.1. In fact, the sequence $\{[n:m]\}_{m \geq n}$ behaves much like $\{M_m\}_{m \geq 1}$. If one takes Proposition 2.2.9(1) and Corollary 2.6.3 and does the substitution $M_m \to [n:m]$ and $e_i \to \varepsilon_i$, one obtains the following result:

Proposition 2.7.10. *The following hold for all* $n, m \in \mathbb{N}$ *where the intervals are well-defined:*

1. $\varepsilon_m x \varepsilon_m = E_{[n:m-2]}(x) \varepsilon_m$ *if* $x \in [n:m-1]$.

2. $E_{[n:m]}(\varepsilon_{m+1}) = \tau$.

The substitution only provides intuition and is not a proof. The proof is an easy application of the Jones relations plus Proposition 2.7.8; it is somewhat similar to the proofs of Proposition 2.2.9(1) and Corollary 2.6.3.

To further study the ε_i and $[n:m]$, we introduce two *graphical* tools: the graphical trace and the trace-preserving reflection. In [Jon83], Jones developed purely algebraic analogues of these tools. By using the graphical Temperley-Lieb algebra, we avoid cumbersome algebraic reasoning.

The graphical trace

TL lacks some of the structure that $[1:\infty]$ has: in particular, it is not equipped with a trace. However, for any $n \geq 0$, the $*$-subalgebra $TL_n := \langle 1, E_1, \ldots E_n \rangle_{\text{alg}}$ can be equipped with a trace, although it cannot extend to all of TL.

Suppose $U \in TL_n$ is a diagram. Let its *closure* be the diagram formed by joining each vertical pair of points as shown below, and let $\mathrm{tr}_n^{\mathrm{gr}}(U)$ be the scalar value of this diagram, i.e. $\mathrm{tr}_n^{\mathrm{gr}}(U) := (\tau^{-1/2})^k$, where k is the number of loops.

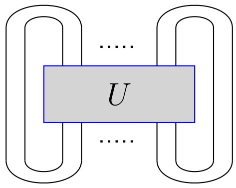
$\mathrm{tr}_n^{\mathrm{g}}(U) =$ 

Figure 2.4: Definition of the graphical trace. (Grey shading indicates U could be any diagram.)

In Figure 2.4, we've joined some pairs of points by strands to the left, and some by strands to the right, but the choice doesn't matter. We could draw U on a sphere and any choice of joinings (as long as strands don't cross) would give isotopic diagrams. Moreover, $\mathrm{tr}_n^{\mathrm{gr}}$ is cyclic as, in a diagram product UV, we can push U along the upper strands until it ends up at the bottom of the diagram. Hence, each $\mathrm{tr}_n^{\mathrm{gr}}$ is well-defined, and a trace.

However, they're not well-behaved traces. The inclusions $\iota_n : TL_n \to TL_{n+1}$ are only trace-preserving up to a scalar multiple[15], so the traces can't be extended to TL. They may also lack the properties we have assumed for $\mathrm{tr} : \mathcal{M} \to \mathbb{C}$. They may not be faithful or positive; in fact, we will later construct explicit counterexamples to positivity. However, they do have a Markov-type property.

Proposition 2.7.11. *For all $j \geq 1$, the following holds for all $x \in TL_j(\delta)$.*

$$\mathrm{tr}_{j+1}^{\mathrm{gr}}(\iota_j(x)E_{j+1}) = \mathrm{tr}_j^{\mathrm{gr}}(x). \tag{2.20}$$

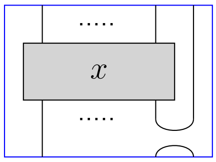
Proof. Assume x is a single diagram. Recall ι_j adds a strand to the right. Then,
The right side is $\mathrm{tr}_j^{\mathrm{gr}}(x)$ by definition of the graphical trace. □

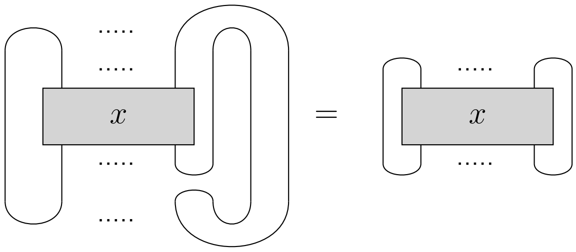
The Markov-type property determines the graphical trace, up to scaling.

Proposition 2.7.12. *Suppose $\{\widetilde{\mathrm{tr}}_j\}_{j=0}^n$ is a family of traces on $\{TL_j(\tau)\}_{j=0}^n$ satisfying* (2.20) *for $0 \leq j \leq n$. Then $\widetilde{\mathrm{tr}}_j = \lambda \,\mathrm{tr}_j^{\mathrm{gr}}$ for $0 \leq j \leq n$, where $\lambda = \widetilde{\mathrm{tr}}_0(1)/\,\mathrm{tr}_0^{\mathrm{gr}}(1)$.*

Proof. Induct on n. $TL_0(\tau)$ is the algebra of single-strand diagrams and hence isomorphic to $\mathbb{C}1$, so the result is immediate. Suppose the result holds for $\{\widetilde{\mathrm{tr}}_j\}_{j=0}^{n-1}$. Then it remains only to show $\widetilde{\mathrm{tr}}_n = \lambda \,\mathrm{tr}_n^{\mathrm{gr}}$.

[15]It is a short exercise to check that $\mathrm{tr}_{n+1}^{\mathrm{gr}}(\iota_n(x)) = \tau^{-1/2}\,\mathrm{tr}_n^{\mathrm{gr}}(x)$.

$\iota_j(x)E_{j+1} =$ 

$\mathrm{tr}^{\mathrm{gr}}_{j+1}(\iota_j(x)E_{j+1}) =$ 

It suffices to show $\widetilde{\mathrm{tr}}_n(W) = \lambda\,\mathrm{tr}^{\mathrm{gr}}_n(W)$ whenever W is a reduced word in $E_1,\dots,E_n$. By Proposition 2.7.8, we can assume W contains one copy of E_n (or else we'd be finished, by induction). Hence $W = UE_nV$ where U,V are words in $E_1,\dots,E_{n-1}$. That means $U,V \in TL_{n-1}$, so we write $U = \iota_{n-1}(U)$. As the trace is cyclic, assume $V = 1$. Then, $W = \iota_{n-1}(U)E_n$.

$$\widetilde{\mathrm{tr}}_n(\iota_{n-1}(U)E_n) = \widetilde{\mathrm{tr}}_{n-1}(U) = \lambda\,\mathrm{tr}^{\mathrm{gr}}_{n-1}(U) = \lambda\,\mathrm{tr}^{\mathrm{gr}}_n(\iota_{n-1}(U)E_n)$$

where the first equality holds as $\widetilde{\mathrm{tr}}_n$ satisfies (2.20), the second holds by induction, and the third holds as $\mathrm{tr}^{\mathrm{gr}}_n$ satisfies (2.20). It follows that $\widetilde{\mathrm{tr}}_n(W) = \lambda\,\mathrm{tr}^{\mathrm{gr}}_n(W)$. □

This is important, because we can equip TL_n with a second trace. Recall from Proposition 2.7.5 that we have a representation $\varphi : TL \to [1:\infty]$. Write $\varphi_j := \varphi|_{TL_j}$. Define the *pullback trace* on TL_j to be the trace $\mathrm{tr}\circ\varphi_j : TL_j \to \mathbb{C}$. Because tr satisfies the Markov property by assumption (Definition 2.7.1), it is an easy calculation to show that the family $\{\tau^{-j/2}\,\mathrm{tr}\circ\varphi_j\}_{j=1}^n$ satisfies the Markov-type property (2.20). We conclude:

Proposition 2.7.13. *The pullback trace is proportional to the graphical trace. Specifically, for all $n \geq 0$, $\mathrm{tr}\circ\varphi_n = \tau^{n/2}\,\mathrm{tr}^{\mathrm{gr}}_n$ on $TL_n(\tau)$.*

This is remarkable – the algebraic *and* tracial structure of the $\{\varepsilon_i\}$ can be described graphically. Conversely, some of the good properties of tr are inherited by the $\mathrm{tr}^{\mathrm{gr}}_n$. (For convenience, we re-state our implicit assumptions.)

Theorem 2.7.14. *If $\{\varepsilon_i\}_{i=1}^\infty \subset \mathcal{M}$ is a family of projections satisfying Jones relations with modulus τ, then $\mathrm{tr}^{\mathrm{gr}}_n$ is a positive trace on $TL_n(\tau)$ for all $n \geq 1$.*

A priori, there's no reason to expect a graphical trace is positive. Although $\mathrm{tr}^{\mathrm{gr}}_n$ takes nonnegative values on diagrams, $\mathrm{tr}^{\mathrm{gr}}_n(x^*x)$ may be negative if x is a linear combination of diagrams. Hence Theorem 2.7.14 is an extremely strong constraint, which we use in Section 2.9 to constrain τ, and hence the index.

Trace-preserving reflection

The next tool we need to prove the index theorem is a 'reflection' map. Let $\Sigma_{1,n} : TL_n \to TL_n$ denote the $*$-automorphism given by reflecting diagrams across a central vertical axis. Clearly, $\Sigma_{1,n}$ maps $E_i \mapsto E_{n+1-i}$. Since the graphical trace of a diagram U is the same whether strands are closed-off to the left or the right (see discussion after Figure 2.4), $\Sigma_{1,n}$ is trace-preserving. In fact, this involution descends to an involution on the algebras $[n:m] = \langle 1, \varepsilon_n, \ldots, \varepsilon_m \rangle$.

Lemma 2.7.15. *Suppose $n \le m$. Then there exists a* tr*-preserving von Neumann algebra automorphism $\sigma_{n,m} : [n:m] \to [n:m]$ mapping $\varepsilon_i \mapsto \varepsilon_{m+n-i}$.*

In [Jon83], a combinatorial argument is used to prove $\sigma_{n,m}$ is well-defined. We circumvent this by defining $\sigma_{1,n}$ as the quotient of $\Sigma_{1,n}$, a map that's *obviously* well-defined as it can be graphically described.

Proof. We may as well $n = 1$. Recall $\varphi_m = \varphi|_{TL_m} : TL_m \to [1:m]$. As this map is surjective, consider it a quotient.

$$\begin{array}{ccc} TL_m & \xrightarrow{\Sigma_{1,m}} & TL_m \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ [1:m] & \overset{\sigma_{1,m}}{\dashrightarrow} & [1:m] \end{array}$$

We claim that $\Sigma_{1,m} : TL_m(\tau) \to TL_m(\tau)$ descends to the quotient. If it does, then it descends to a map $\sigma_{1,m}$ sending $\varepsilon_i \mapsto \varepsilon_{m+1-i}$, as required.

We need to show that $\varphi(x) = 0 \implies \varphi(\Sigma_{1,m}(x)) = 0$. From Proposition 2.7.13, $\mathrm{tr} \circ \varphi_m$ is proportional to the graphical trace, and $\Sigma_{1,m}$ preserves the graphical trace, so $\mathrm{tr}(\varphi(x)) = \mathrm{tr}(\varphi(\Sigma_{1,m}(x)))$ for all $x \in TL_m$.

In particular, if $\varphi(x) = 0$, then $0 = \mathrm{tr}(\varphi(x^*x)) = \mathrm{tr}(\varphi(\Sigma_{1,m}(x^*x)))$. As tr is faithful, $\varphi(\Sigma_{1,m}(x^*x)) = 0$. This argument shows $\sigma_{1,m}$ is well-defined, and automatically trace-preserving because $\Sigma_{1,m}$ is. $\square$

In Definition 2.7.1, it's clear that relations 1-3 (the algebraic Jones relations) are symmetric under reversing the order of indices. The existence of the trace-preserving reflection means *tracial structure* is also left intact by this reversal. For example, the Markov property (Definition 2.7.1(4)) is preserved under reflection.

Lemma 2.7.16. *(Reflected Markov property)*
$\operatorname{tr}(x\varepsilon_i) = \tau \operatorname{tr}(x)$ *if* $x \in [i+1:\infty]$.

Proof. Assume x is a word in ε_j for $j \geq i+1$. As words have finite length, $w \in [i+1:m]$ for some $m \in \mathbb{N}$. Reflect the left side, apply the usual Markov property, and reflect back: $\operatorname{tr}(x\varepsilon_i) = \operatorname{tr}(\sigma_{i,m}(x)\varepsilon_m) = \tau \operatorname{tr}(\sigma_{i,m}(x)) = \tau \operatorname{tr}(x)$. □

Because $\sigma_{1,m}$ preserves tracial structure, it respects conditional expectations[16], i.e. $E_{[n:m]} = \sigma_{1,m} \circ E_{[1:m-n+1]} \circ \sigma_{1,m}$. Then, one can take Proposition 2.7.10 and 'reflect the indices'. E.g. to obtain the first result below, take Proposition 2.7.10(1) and send $n \to m$, $n+1 \to m-1$, etc.

Proposition 2.7.17. *The following hold for all finite* $n, m \in \mathbb{N}$ *where the intervals are nontrivial:*

1. $\varepsilon_n x \varepsilon_n = E_{[n+2:m]}(x)\varepsilon_n$ *if* $x \in [n+1:m]$.

2. $E_{[n:m]}(e_{n-1}) = \tau$.

An key difference between Proposition 2.7.10 and its reflected counterpart Proposition 2.7.17 is that the intervals above can have arbitrarily large right bounds. Since we want to construct II_1 factors from the $[n:m]$, and II_1 factors are infinite-dimensional, we will need to take these right bounds to infinity. To do so, we need to know how to compute conditional expectations like $E_{[n:\infty]''}$.

Recall $[n:m]$ is the unital $*$-subalgebra generated by $\{\varepsilon_i\}_{i=n}^m$ and $[n:m]''$ is the von Neumann algebra generated by $\{\varepsilon_i\}_{i=n}^m$. They are distinct when $m = \infty$.

Proposition 2.7.18. $E_{[n:m]} \xrightarrow[m\to\infty]{\text{SOT}} E_{[n:\infty]''}$ *on* $[n-1:\infty]''$.

This result is stated in [Jon83, 4.1.12].

Proof. By relabelling indices, assume $n = 2$. First, we'll prove that the sequence $\{E_{[2:m]}(x)\}_{m\geq 2}$ eventually stabilises for x in the dense subspace $[1:\infty]$. As $[1:\infty]$ is spanned by words in $\varepsilon_1, \varepsilon_2, \ldots$, assume x is a word. As words are finite, x is a word in $\varepsilon_1, \ldots, \varepsilon_{m_0}$ for some $m_0 \in \mathbb{N}$. By Proposition 2.7.8, we can assume it contains exactly one copy of ε_k where k is minimal among indices appearing in x. That is, $x = x_1\varepsilon_k x_2$ for $x_1, x_2 \in [k+1:m_0] \subset [2:m_0]$. Then, for all $m \geq m_0$,

[16] By Proposition 2.2.7, a trace-preserving conditional expectation is uniquely determined by the trace.

$$E_{[2:m]}(x) = E_{[2:m]}(x_1\varepsilon_k x_2) = x_1 E_{[2:m]}(\varepsilon_k)x_2 = \begin{cases} x_1\varepsilon_k x_2 & \text{if } m_0 \geq k \geq 2. \\ \tau x_1 x_2 & \text{if } k = 1. \end{cases}$$

where the last equality holds by Proposition 2.7.17(2). The above expression for $E_{[2:m]}(x)$ is independent of m for $m \geq m_0$, so the sequence stabilises as claimed.

It is easy to show that conditional expectations have operator norm at most 1, so $\{E_{[2:m]}\}_{m\geq 2}$ is a norm-bounded sequence of operators converging strongly on a dense subset $[1:\infty]$ of $[1:\infty]''$. Note that $E_{[2:m]}$ maps $[1:\infty]''$ to $[2:m]$, so the sequence as a whole can be defined as operators $[1:\infty]'' \to [2:\infty]''$. A standard functional analysis fact implies that the sequence converges strongly everywhere to a bounded operator $F : [1:\infty]'' \to [2:\infty]''$.

We claim $F = E_{[2:\infty]''}$ on $[1:\infty]''$. It suffices to check the equivalent characterisation of conditional expectations from Proposition 2.2.7, i.e. that $\mathrm{tr}(F(x)y) = \mathrm{tr}(xy)$ for all $x \in [1:\infty]''$, $y \in [2:\infty]''$. Indeed, it suffices to check x, y in the dense subsets $[1:\infty]$ and $[2:\infty]$ respectively[17]. Let $x \in [1:i]$ for large i and $y \in [2:j]$ for large j. For all $k \geq j$, $y \in [2:k]$, so Proposition 2.2.7 implies $\mathrm{tr}(E_{[2:k]}(x)y) = \mathrm{tr}(xy)$. Taking $k \to \infty$, we find that $\mathrm{tr}(F(x)y) = \mathrm{tr}(xy)$. Hence $E_{[2:\infty]''} = F = \text{s-lim}_{m\to\infty} E_{[2:m]}$ on the requisite domain. □

Armed with this result, we can take the identities from Proposition 2.7.17 and take the limit as the right bounds go to infinity.

Proposition 2.7.19. *For all $n \in \mathbb{N}$,*

1. $\varepsilon_n x \varepsilon_n = E_{[n+2:\infty]''}(x)\varepsilon_n$ *if* $x \in [n+1:\infty]$.

2. $E_{[n:\infty]}(e_{n-1}) = \tau$.

This concludes our preparation for the Jones index theorem. One key outcome is the above, Proposition 2.7.19. It lets us work with infinitely-generated intervals of the form $[m:\infty]$, from which we will build subfactors of index values $4\cos^2(\pi/n)$ in Section 2.8. The other key outcome is Theorem 2.7.14, which gives a condition for the entire tower $\{TL_j(\tau)\}$ to have positive graphical traces. We use this in Section 2.9 this to rule out τ-values, and hence rule out index values.

[17] Because the trace, and F, are normal and hence ultraweakly continuous.

2.8 The Jones index theorem: constructing a subfactor

We now prove the Jones index theorem. One direction of the theorem asserts the existence of subfactors of certain indices.

Theorem 2.8.1. *(Jones index theorem – existence)*

For every value in $\{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty]$, *there exists a* II_1 *subfactor with index equal to that value.*

We follow the proof of [Jon83].

The construction of a subfactor $N \subset M$ with $[M : N] = \infty$ can be done with standard techniques; see [Jon09, 19.2.10]. For the finite values, the proof separates into two cases: construction of subfactors with index in $[4, \infty)$, and construction of subfactors with index $4\cos^2(\pi/n)$.

The former is basically a classical construction and does not rely on the basic construction or any other similarly powerful tools in subfactor theory except, of course, for the notion of index. We sketch an argument. In their fourth paper on von Neumann algebras, Murray and von Neumann [MN43, pp781-784] constructed a II_1 factor R known as the *hyperfinite* II_1 *factor* which, among many remarkable properties, satisfies $pRp \cong R$ whenever $p \in R$ is nonzero (i.e., R is isomorphic to a corner of itself).

Let $\alpha \geq 4$. To exhibit a subfactor of index α, let $d \in (0, 1)$ be such that $\frac{1}{d} + \frac{1}{1-d} = \alpha$. By Proposition 1.7.3, there exists a projection $p \in R$ with trace d. By the above remark, $pRp \cong R \cong (1-p)R(1-p)$. Let $\theta : pRp \to (1-p)R(1-p)$ be an explicit isomorphism. Set $S = \{x + \theta(x) : x \in pRp\}$. As $(pRp)((1-p)R(1-p)) = 0$, multiplication in S is termwise with respect to the terms $x, \theta(x)$. Consequently, if $z + \theta(z) \in Z(S)$, then $z \in Z(pRp) = \mathbb{C}1$. Hence S is a factor and hence a II_1 factor because it inherits a trace. Evidently $pSp = pRp$, and $(1-p)S(1-p) = \theta(pRp) = (1-p)R(1-p)$. Proposition 1.8.3(4) gives a decomposition of the index as follows:

$$\begin{aligned}[R : S] &= \mathrm{tr}_R(p)^{-1}[pRp : pSp] + \mathrm{tr}_R(1-p)^{-1}[(1-p)R(1-p) : (1-p)S(1-p)] \\ &= \frac{1}{d} + \frac{1}{1-d} = \alpha.\end{aligned}$$

Of course, this construction fails to exhibit a II_1 subfactor for $\alpha < 4$ as $\frac{1}{d} + \frac{1}{1-d}$ is minimised at 4 for $d \in (0, 1)$.

In marked contrast to the $\alpha \geq 4$ case, the construction of subfactors with index $4\cos^2(\pi/n)$ belongs decidedly to modern subfactor theory; it relies integrally on the theory of Jones towers and Jones relations. We will show that a family $\{\varepsilon_i\}_{i=1}^{\infty}$ with Jones relations of modulus τ gives rise to a subfactor of index τ^{-1}. (The proof of Theorem 2.8.1 concludes after Corollary 2.8.4.)

Recall that $[n : m]$ is the $*$-algebra generated by $1, \varepsilon_n, \ldots, \varepsilon_m$. Using the ε_i, we build an inclusion of infinite-dimensional tracial von Neumann algebras. The easiest choice is $[2 : \infty]'' \subset [1 : \infty]''$, as they differ by just one generator.

Theorem 2.8.2. *Suppose $\{\varepsilon_i\}_{i=1}^{\infty}$ is a family of projections satisfying the Jones relations with modulus $\tau > 0$. Let $P = [1 : \infty]''$ and $P^{(\tau)} = [2 : \infty]''$. Then $P^{(\tau)} \subset P$ is a II_1 subfactor with index τ^{-1}.*

There are two distinct facts here – that $P, P^{(\tau)}$ are II_1 factors, and that the index is τ^{-1}. The first has a number of proofs. One is a lengthy but explicit computation showing their centres are trivial, found in [GHJ89, 3.4.4]. Jones's original argument [Jon83, 4.1.9] invokes ergodic theory and is beyond our scope. We will take for granted that $P, P^{(\tau)}$ are II_1 factors.

What is really important to the proof of the index theorem is the value of the index $[P : P^{(\tau)}]$. To prove it is τ^{-1}, we can form the basic construction $P_1 = \langle P, e_{P^{(\tau)}} \rangle$. Then, by Proposition 2.4.4, we can compute the index with the formula $\mathrm{tr}(e_{P^{(\tau)}}) = [P : P^{(\tau)}]^{-1}$. But first, we must understand what P_1 looks like. (The proof of Theorem 2.8.2 concludes after the proof of Proposition 2.8.3.)

P has a nice generating set[18] $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$, whose elements satisfy the Jones relations. P_1 has the generating set $\{e_{P^{(\tau)}}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\}$. Remarkably, $e_{P^{(\tau)}}$ is compatible with the Jones relations. Namely, if we set $\varepsilon_0 = e_{P^{(\tau)}}$, then the set $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots\}$ *still* satisfies the Jones relations. In other words, the basic construction simply extends the Jones relations downwards by a step.

Proposition 2.8.3. *If $\varepsilon_0 := e_{P^{(\tau)}}$, then $\{\varepsilon_i\}_{i=0}^{\infty}$ satisfy the Jones relations with modulus τ.*

Pieces of this result are scattered across [Jon83]; we gather them in one proof.

Proof. Since $\{\varepsilon_i\}_{i=1}^{\infty}$ already satisfy the Jones relations, it remains to prove the algebraic Jones relations involving ε_0:

1. $\varepsilon_0^2 = \varepsilon_0$.

[18] This set generates P *as a von Neumann algebra.* Note that 1 can always be omitted from a generating set, by Definition 1.2.6.

2. $\varepsilon_0\varepsilon_1\varepsilon_0 = \tau\varepsilon_0$ and $\varepsilon_1\varepsilon_0\varepsilon_1 = \tau\varepsilon_1$.

3. $\varepsilon_0\varepsilon_i = \varepsilon_0\varepsilon_i$ if $i \geq 2$.

as also the Markov property, which has to be modified to account for the presence of ε_0:

4. $\operatorname{tr}(x\varepsilon_i) = \tau \operatorname{tr}(x)$ if $w \in [0 : i]$.

The first relation is immediate as ε_0 is a projection. The third relation is immediate as $\varepsilon_i \in P^{(\tau)}$ for $i \geq 2$, and, by Proposition 2.2.9(2), $\varepsilon_0 = e_{P^{(\tau)}}$ commutes with $P^{(\tau)}$. The second relation is more nontrivial. To prove the first part:

$$\varepsilon_0\varepsilon_1\varepsilon_0 = e_{P^{(\tau)}}\varepsilon_1 e_{P^{(\tau)}} = E_{P^{(\tau)}}(\varepsilon_1)e_{P^{(\tau)}} = E_{[2:\infty]''}(\varepsilon_1)e_{P^{(\tau)}} = \tau e_{P^{(\tau)}} = \tau\varepsilon_0.$$

where the second equality is by Proposition 2.2.9(1), and the second from the right is by Proposition 2.7.19(2). (This is where we needed to invoke our technical facts about 'infinite intervals'.) The second part is more involved, but similar.

It remains to check the Markov property. We note that, because the trace-preserving reflection exists, it suffices to verify the reflected Markov property $\operatorname{tr}(x\varepsilon_i) = \tau \operatorname{tr}(x)$ for $x \in [i+1 : \infty]$. This is advantageous, because this is already true for $i \geq 1$, due to Lemma 2.7.16. We only need to prove the equality in the case where $i = 0$, i.e. $\operatorname{tr}(xe_{P^{(\tau)}}) = \tau \operatorname{tr}(x)$ for $x \in P$. This tautologically the statement that $P^{(\tau)} \subset P \subset P_1$ is Markov with modulus τ (Definition 2.3.2).

We will show that $P^{(\tau)} \subset P \subset P_1$ is Markov with modulus $[P : P^{(\tau)}]^{-1}$, and then show that the moduli agree. By the existence-uniqueness condition from Theorem 2.4.6, if $P^{(\tau)} \subset P$ has finite index, then $P^{(\tau)} \subset P$ has a Jones tower with modulus $[P : P^{(\tau)}]^{-1}$, which implies in particular that $P^{(\tau)} \subset P \subset P_1$ is Markov with modulus $[P : P^{(\tau)}]^{-1}$. Hence we only need to show that $P^{(\tau)} \subset P$ has finite index. But recall, if $(P^{(\tau)})'$ is the commutant of $P^{(\tau)}$ on $L^2(P)$, then

$$[P : P^{(\tau)}] < \infty \iff (P^{(\tau)})' \text{ is finite} \iff P_1 \text{ is finite.}$$

where the first equivalence is a basic fact about the index (Proposition 1.8.3) and the latter holds as $(P^{(\tau)})'$ is antilinearly isomorphic to P_1 (Proposition 2.2.15).

Hence we need only show that the identity in P_1 is finite. Although P_1 contains a II_1 factor P, which is finite, finiteness in a sub-von-Neumann algebra does not imply finiteness in the containing algebra. In fact, exhibiting just one nonzero projection that is finite in both P and P_1 will force P_1 to be finite.

Because we already proved the second Jones relation, we have that $\varepsilon_0\varepsilon_1\varepsilon_0 = \tau\varepsilon_0$ and $\varepsilon_1\varepsilon_0\varepsilon_1 = \tau\varepsilon_1$. If we set $u = \tau^{-1/2}\varepsilon_1\varepsilon_0$, then $uu^* = \varepsilon_1$ and $u^*u = \varepsilon_0$, and hence ε_1 and ε_0 are Murray-von Neumann equivalent. We have that $\varepsilon_0 = e_{P^{(\tau)}}$ is finite in P_1 by Corollary 2.2.13, and hence ε_1 must also be finite in P_1 as equivalence preserves finiteness. Hence ε_1 is finite in both P and P_1. We claim that this means P_1 is finite.

The following argument is from [GHJ89, p155]. To show 1 is finite in P_1, we'll decompose it as $1 = \sum_{i=1}^{d} q_i$ where each $q_i \in P$ and is finite in P_1.

To show q_i is finite, the trick is to show that $\mathrm{tr}(q_i) \leq \mathrm{tr}(\varepsilon_1)$. As P is a II_1 factor, this implies (by Proposition 1.7.3) that $q_i \preceq \varepsilon_1$ in P. Unlike finiteness, Murray-von Neumann ordering is preserved under inclusions, so $q_i \preceq \varepsilon_1$ in P_1 as well, and hence the q_i are finite because ε_1 is.

Let $d \in \mathbb{N}$ be such that $\mathrm{tr}(\varepsilon_1) > 1/d$. By Proposition 1.7.3, as P is a II_1 factor, it contains a projection q_1 with $\mathrm{tr}(q_1) = 1/d \leq \mathrm{tr}(\varepsilon_1)$. By Lemma 1.4.4, $(1-q_1)P(1-q_1)$ is a factor and hence a II_1 factor[19]. We can hence find a projection q_2 in it whose trace in P is $1/d$. Then do the same for $(1-q_1)(1-q_2)P(1-q_2)(1-q_1)$ and so on, to obtain mutually orthogonal projections $q_1, \ldots, q_d$, where each q_i has trace $\mathrm{tr}(q_i) = 1/d \leq \mathrm{tr}(\varepsilon_1)$. Hence, as explained in the above paragraph the q_i are all finite in P_1. Moreover, $\sum_{i=1}^{d} q_i = 1$, which implies 1 is finite in P_1.

We are done, so long as we can show that the Markov moduli agree, i.e. $\tau = [P : P^{(\tau)}]^{-1}$. But equivalent projections have the same trace, so $\tau = \mathrm{tr}(\varepsilon_1) = \mathrm{tr}(\varepsilon_0) = \mathrm{tr}(e_{P^{(\tau)}}) = [P : P^{(\tau)}]^{-1}$, where the first is by Definition 2.7.1, and the last is by Proposition 2.4.4. □

We have the formula $[P : P^{(\tau)}] = \tau^{-1}$, concluding the proof of Theorem 2.8.2. But the formula is just one consequence of a profound result: the basic construction attaches a projection ε_0 to $\{\varepsilon_1, \varepsilon_2, \ldots\}''$ while seamlessly maintaining the Jones relations. Hence we write $P_1 = [0 : \infty]''$. Of course, the phenomenon remains true if the basic construction is iterated.

Corollary 2.8.4. *As usual, let $\{P_n\}_{n=-1}^{\infty}$ denote the Jones tower of $P^{(\tau)} \subset P$ and $e_n \in P_n$ be the Jones projections. If we let $\varepsilon_j := e_{1-j} \in P_{1-j}$, then $\{\varepsilon_i\}_{i=-\infty}^{\infty}$ satisfy the Jones relations with Markov modulus τ. Hence $P_i = [1-i : \infty]''$.*

In short: the Jones tower of $P^{(\tau)} \subset P$ extends the sequence $\{\varepsilon_1, \varepsilon_2, \ldots\}$ downwards to negative infinity. We use this later; for now, the important result is the formula $[P : P^{(\tau)}] = \tau^{-1}$. This shows that we can control the index of $P^{(\tau)} \subset P$, a quantity that's *a priori* hard to compute, just by tuning an algebraic parameter.

[19]Because it inherits a trace.

Recall that we already constructed subfactors with index four or greater, so the last step to prove Theorem 2.8.1 is to construct subfactors of index $4\cos^2(\pi/n)$.

Proof. (Theorem 2.8.1)

As remarked after Theorem 2.5.11, for all $n \geq 3$, because the Coxeter diagram A_{n-1} has norm-squared $4\cos^2(\pi/n)$, it induces a Jones tower of finite-dimensional algebras, $\{B_i\}_{i=-1}^{\infty}$, with modulus $\tau = 1/(4\cos^2(\pi/n))$. By definition of the Jones tower (Definition 2.3.1), the tower contains a sequence $\{e_i\}_{i\in\mathbb{N}}$. By Theorem 2.7.3, if we set $\varepsilon_i := e_i$, then $\{e_i\}_{i\in\mathbb{N}}$ satisfy the Jones relations with modulus τ. Hence we can apply Theorem 2.8.2 to obtain the subfactor $P^{(\tau)} \subset P$, which has index $[P : P^{(\tau)}] = \tau^{-1} = 4\cos^2(\pi/n)$. □

This establishes that every index value appearing in the Jones index theorem is indeed realised by a subfactor. It remains to constrain the index to those values.

Before we do this in the following section, we make an observation. If one had a II_1 subfactor $N \subset M$ to begin with, then one could substitute the tower $\{B_i\}$ used above for the Jones tower $\{M_i\}_{i=-1}^{\infty}$ of $N \subset M$. As its Markov modulus is $[M : N]^{-1}$, the procedure would return a subfactor $P^{([M:N]^{-1})} \subset P$ with index $[M : N]$. It is *not*, in general, isomorphic to the subfactor one started with.

In fact, if $\tau^{-1} = 4\cos^2(\pi/n)$ for $n \geq 3$, the subfactor $P^{(\tau)} \subset P$ depends *only* on n [Pop90, 6.7] [Pop94, p231]. That is, no matter how one obtains a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ satisfying Jones relations with modulus $1/(4\cos^2(\pi/n))$, it produces the same subfactor, up to isomorphism. E.g. one could choose a different graph with norm-squared $4\cos^2(\pi/n)$ [GHJ89, 1.4.3]. We give this subfactor a name.

Definition 2.8.5. Let $n \geq 3$, and let $\{\varepsilon_i\}_{i=1}^{\infty}$ be any family of projections satisfying Jones relations with modulus $1/(4(\cos^2(\pi/n)))$. Define the *Jones subfactor* $J^{(n)} \subset J$ by $J = [1 : \infty]''$, $J^{(n)} = [2 : \infty]''$.

We know $J^{(n)} \subset J$ has index $4\cos^2(\pi/n)$, but not much else. In Chapter 3, we will embark on more extensive study. For now, we prove the other half of the index theorem, which requires us to constrain the values of the index.

2.9 The Jones index theorem: constraining the index

The proof in this section is essentially due to Hans Wenzl [Wen87]; see also [Pen14, pp13-15].

Theorem 2.9.1. *(Jones index theorem - constraint on the index)*
If $N \subset M$ is a II_1 subfactor, then

$$[M : N] \in \{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty]. \tag{2.21}$$

The point of developing the theory of the Jones/Temperley-Lieb relations is to reduce questions about subfactors to questions about the relations. If $N \subset M$ is a finite index II_1 subfactor, then it has a Jones tower of modulus $[M : N]^{-1}$, and the projections $\{e_i\}_{i\in\mathbb{N}}$ satisfy Jones relations. Theorem 2.7.14 then implies that the graphical trace $\mathrm{tr}_j^{\mathrm{gr}} : TL_j([M : N]^{-1}) \to \mathbb{C}$ is positive for all $j \in \mathbb{N}$. To prove Theorem 2.9.1, it hence suffices to prove the following:

Theorem 2.9.2. *If all graphical traces in the tower $\{TL_j(\tau)\}_{j\in\mathbb{N}}$ are positive, then $\tau^{-1} \in \{4\cos^2(\pi/n) : n = 3, 4, 5, \ldots\} \cup [4, \infty)$.*

(The proof concludes after Lemma 2.9.5.) This is a substantially more approachable task. To rule out a value of τ, we need to exhibit just one nonpositive graphical trace $\mathrm{tr}_m^{\mathrm{gr}}$. To prove $\mathrm{tr}_m^{\mathrm{gr}}$ is nonpositive, we need to exhibit just one positive element $f = x^*x \in TL_m(\tau)$ such that $\mathrm{tr}_m(f) < 0$. We do this by constructing (further below, in Definition 2.9.4) a finite or infinite sequence $f^{(0)}, \ldots, f^{(j)}, \ldots$ of nontrivial projections, where $f^{(j)} \in TL_j(\tau)$ for all j.

As a preliminary to the proof, we introduce the notion of a quantum integer.

Definition 2.9.3. Suppose $q \in \{e^{i\theta} : \theta \in (0, \pi/2)\} \cup [1, \infty)$. Then, for $n \in \{0, 1, 2, \ldots\}$, the *quantum integer* $[n]_q$ is defined by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{-(n-1)} \tag{2.22}$$

where we take $[n]_1 := n$.

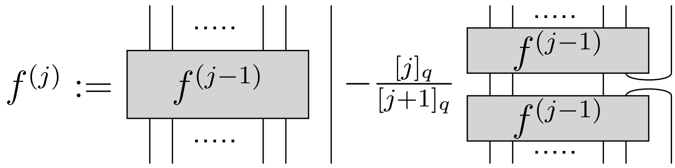
We re-parameterise the 'loop value' $\tau^{-1/2}$ by choosing q so that $\tau^{-1/2} = [2]_q$. Specifically, let $q = q(\tau)$ be the positive (if real) or principal (if complex) root of $0 = q^2 - (\tau^{-1/2})q + 1$. One easily checks that q lies in the domain given above and that $[2]_q = q + q^{-1} = \tau^{-1/2}$. By checking the discriminant, we see that, if $\tau^{-1} < 4$, then q is in the complex part of its domain, while if $\tau^{-1} \geq 4$, then $q \geq 1$. Some useful formulae are as follows [Mor17, p1] [Pen14, pp14-15]:

$$[m]_q[n+1]_q - [m-n]_q = [m+1]_q[n]_q \text{ whenever } n \leq m. \tag{2.23}$$

$$[n]_q = \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(n\theta)}{\sin(\theta)} \text{ when } q \in \{e^{i\theta} : \theta \in (0, \pi/2)\}. \tag{2.24}$$

We define the sequence alluded to above as follows:

Definition 2.9.4. Define the 0th *Jones-Wenzl idempotent* $f^{(0)}$ to be the identity, or single-strand diagram, in $TL_0([2]_q)$. Inductively for $j \geq 1$, if $f^{(j-1)}$ is defined and $[j+1]_q$ is nonzero, then define the jth Jones-Wenzl idempotent, $f^{(j)} \in TL_j([2]_q)$, to be the following linear combination of diagrams:



This definition is essentially due to Hans Wenzl [Wen87], although the pictorial formula came later (see, e.g., [Pen14]). One can check inductively that these are indeed idempotents (in fact, projections), i.e. $\left(f^{(j)}\right)^2 = f^{(j)}$.

It's clear from this definition that, if j is the minimal integer such that $[j]_q = 0$, then $f^{(j-2)}$ is the highest Jones-Wenzl idempotent that can be defined (for that particular choice of q).

We compute the sequence $\mathrm{tr}_0^{\mathrm{gr}}(f^{(0)}), \mathrm{tr}_1^{\mathrm{gr}}(f^{(1)}), \ldots$ to see if it contains a negative number, which would be a counterexample to positivity. There is a nice formula:

Lemma 2.9.5. *If $j \geq 0$ is such that $f^{(j)}$ is defined, then $\mathrm{tr}_j(f^{(j)}) = [j+2]_q$.*

Proof. Proceed by induction. $\mathrm{tr}_0^{\mathrm{gr}}(f^{(0)}) = \mathrm{tr}_0^{\mathrm{gr}}(1) = \tau^{-1/2} = [2]_q$, the loop value. For $j \geq 1$, assume $f^{(j)}$ is defined and assume $\mathrm{tr}_{j-1}^{\mathrm{gr}}(f^{(j-1)}) = [j+1]_q$. Compute the trace of $f^{(j)}$ by 'closing off' the diagrams, as usual; see Figure 2.5.

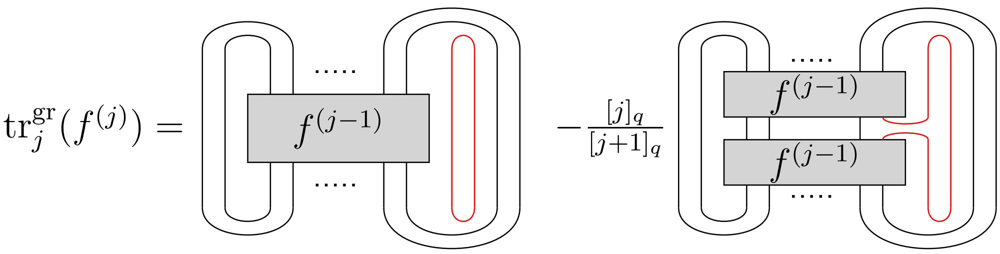


Figure 2.5: Computing the trace of a Jones-Wenzl idempotent.

The left-hand diagram has the value $[2]_q \, \mathrm{tr}_{j-1}^{\mathrm{gr}}(f^{(j-1)})$, owing to the extra loop (in red). By shrinking the red strand in the right-hand diagram, we realise that it has the value $\mathrm{tr}(f^{(j-1)} f^{(j-1)}) = \mathrm{tr}(f^{(j-1)})$, as $f^{(j)}$ is an idempotent. Hence,

$$\mathrm{tr}_j^{\mathrm{gr}}(f^{(j)}) = \left([2]_q - \frac{[j]_q}{[j+1]_q}\right) \mathrm{tr}_{j-1}^{\mathrm{gr}}(f^{(j-1)}) = [2]_q[j+1]_q - [j]_q$$

by the inductive assumption. By (2.23), $\mathrm{tr}_j^{\mathrm{gr}}(f^{(j)}) = [j+2]_q[1]_q = [j+2]_q$. $\square$

We use the $f^{(j)}$ to find counterexamples to positivity of graphical traces.

Proof. (Theorem 2.9.2)

As the values of τ^{-1} that we want to reject lie between 0 and 4, assume that $\tau^{-1} < 4$. Recall $\tau^{-1/2} = [2]_q$, and recall from the discussion after Definition 2.9.3 that $\tau^{-1} < 4$ is equivalent to $q \in \{e^{i\theta} : \theta \in (0, \pi/2)\}$. By Lemma 2.9.5, and (2.24), we find that the sequence of traces can be expressed three ways:

$$\mathrm{tr}^{\mathrm{gr}}_{j-2}(f^{(j-2)}) = [j]_q = \frac{\sin(j\theta)}{\sin(\theta)}. \tag{2.25}$$

Let $m \in \mathbb{N}$ be minimal such that $\sin(m\theta) \leq 0$. By (2.25), $[j]_q > 0$ for $j \leq m-1$, meaning that the sequence can be defined at least up to $\mathrm{tr}^{\mathrm{gr}}_{m-2}(f^{(m-2)})$, by Definition 2.9.4.

If $\sin(m\theta) = 0$, then $[m]_q = 0$, so the sequence can't be continued any further. Because $\mathrm{tr}^{\mathrm{gr}}_{j-2}(f^{(j-2)}) = \sin(j\theta)/\sin(\theta) \geq 0$ for $j \leq m$, this means that we fail to find a counterexample to positivity before the sequence terminates.

However, if $\sin(m\theta) < 0$, then $\mathrm{tr}^{\mathrm{gr}}_{m-2}(f^{(m-2)}) < 0$, which *is* a counterexample.

If $\pi/\theta \in \mathbb{N}$, then it's clear that $m = \pi/\theta$, and $\sin(m\theta) = 0$, so we obtain no counterexample. If $\pi/\theta \notin \mathbb{N}$, we claim we obtain one. Set $m = \lfloor \pi/\theta \rfloor + 1$.

$$\begin{aligned} 0 < m - 1 < \frac{\pi}{\theta} < m < \frac{2\pi}{\theta} \\ \implies 0 < (m-1)\theta < \pi < m\theta < 2\pi. \end{aligned}$$

and therefore, as required, $\sin(j\theta) > 0$ for $j \leq m-1$, whereas $\sin(m\theta) < 0$. Hence we obtain a counterexample to positivity of the graphical trace when $\pi/\theta \notin \mathbb{N}$. But, by (2.24), $\tau^{-1} = [2]_q^2 = (\sin(2\theta)/\sin(\theta))^2 = 4\cos^2(\theta)$. Thus the constraint $\pi/\theta \notin \mathbb{N}$ is equivalent to the constraint $\tau^{-1} \neq 4\cos^2(\pi/n)$, as required. □

This proves that, if all graphical traces in the tower $\{TL_j(\tau)\}_{j\geq 0}$ are positive, then τ^{-1} is confined to $\{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty]$. As explained after the statement of Theorem 2.9.1, a subfactor of index $[M : N]$ induces a tower $\{TL_j([M : N]^{-1})\}_{j\geq 0}$ where each graphical trace is positive. Thus, as an *immediate* corollary of the above proof, $[M : N]$ must lie in $\{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty]$.

As we have already constructed subfactors of indices in the allowable set, we obtain our hard-won reward: the Jones index theorem.

Theorem 2.9.6. *(Jones index theorem)*

$$\{[M : N] : N \subset M \textit{ is a } II_1 \textit{ subfactor.}\} = \{4\cos^2(\pi/n) : n \geq 3\} \cup [4, \infty].$$

2.10 Conclusion

This concludes our treatment of the Jones index theorem. We have both constrained the index to the set of allowable values, and demonstrated the existence of a subfactor with each of those index values.

It is remarkable that the Jones/Temperley-Lieb relations play a key role in both directions of the proof. In retrospect, the index theorem can be viewed as a hybrid theorem in subfactor/Temperley-Lieb theory.

However, the most enduring contribution of [Jon83] to subfactor theory is the basic construction and the Jones tower. We have seen that the index is nothing more than a quantity which encodes the Markov modulus – i.e., the minimal amount of information about the tower. In the following chapter, we address the extraction of stronger invariants from the Jones tower.

Chapter 3

The principal graph

3.1 Introduction

'Index for Subfactors' [Jon83] introduced the index and Jones tower of a II_1 subfactor $N \subset M$, setting the foundations of modern II_1 subfactor theory. We witnessed the value of the Jones tower – it unfolds emergent algebraic structure (e.g. the Jones relations) from $N \subset M$. However, it is an infinite tower of infinite-dimensional objects and is generally extremely difficult to compute[1]. It is hence natural to ask: "Can we extract more tractable invariants from the Jones tower?"

In this chapter, we introduce two major invariants which appeared within a decade of [Jon83], and, together with the index, form the three most important invariants[2] of a subfactor.

The first is the *standard invariant* [Ocn88] [GHJ89] [Pop95]. Given a II_1 subfactor $N \subset M$, the standard invariant is formed by intersecting the Jones tower with commutants – it consists of the pair of towers $\{N' \cap M_n\}_{n\geq -1}$ and $\{M' \cap M_n\}_{n\geq 0}$. Remarkably, these towers consist of finite-dimensional algebras, and are hence substantially easier to study than the Jones tower.

From the standard invariant, we construct another invariant: a pair of graphs known as the *principal graph* Γ and *dual principal graph* Γ'. We prove the non-trivial fact that, *if* the principal graphs happen to be finite, then they generalise the index via the relation $\|\Gamma\|^2 = [M : N] = \|\Gamma'\|^2$. Hence, by applying graph theory techniques, we obtain a second proof of the Jones index theorem that is independent of the first.

[1] Except for very simple subfactors such as the Jones subfactor $J^{(n)} \subset J$ of Definition 2.8.5.

[2] Note we don't call the Jones tower an invariant for a subfactor, because it is an *equivalent object* to a subfactor (as it is unique for $N \subset M$, and $N \subset M$ can be recovered from it).

We also leverage these invariants to better understand the Jones subfactor $J^{(n)} \subset J$ with index $4\cos^2(\pi/n)$, given in Definition 2.8.5. We compute the standard invariant and principal graph for this subfactor. These computations are well-known to experts; we give accessible proofs. By the end of this chapter, we will understand nearly all there is to know about this subfactor.

This chapter draws most heavily on the monograph of Goodman, de la Harpe and Jones [GHJ89], a paper of Sorin Popa [Pop90], and later writings by Dietmar Bisch [Bis97].

3.2 The standard invariant

Our intuition suggests that invariant objects are often built from von Neumann algebras and their commutants. For example, the centre of M can be expressed as $Z(M) = M' \cap M$. Then, a natural family of objects to consider are the *relative commutants* $M_n' \cap M_m$ for $n, m \in \{-1, 0, \ldots,\}$. Note that, although commutants are spatial, relative commutants are well-defined independently of representation, as $M_n' \cap M_m$ can be written purely abstractly as $M_n' \cap M_m = \{x \in M_m : xy = yx \text{ for all } y \in M_n.\}$. In fact, these relative commutants are finite-dimensional.

Proposition 3.2.1. *Suppose A, B are II_1 factors such that $[B : A] < \infty$. Then $A' \cap B$ is finite-dimensional with $\mathbb{C}$-dimension* $\dim A' \cap B \le [B : A]$.

Proof. This proof follows [Jon83, 2.2.3], [GHJ89, 3.6.2]. Suppose $\{p_i\}_{i=1}^k \subset A' \cap B$ is a partition of unity, i.e. the p_i are mutually orthogonal and $\sum_i p_i = 1$. We will show $\{p_i\}$ cannot be arbitrarily large.

By Proposition 1.8.3(4), $[pBp : Ap] = \mathrm{tr}_{A'}(p)\,\mathrm{tr}_B(p)[B : A]$ for any projection $p \in A' \cap B$. As $\sum_i \mathrm{tr}_{A'}(p_i) = 1$, we can decompose the index:

$$[B : A] = \sum_{i=1}^{k} (\mathrm{tr}_B(p_i))^{-1} [p_i B p_i : A p_i] \ge \sum_{i=1}^{k} (\mathrm{tr}_B(p_i))^{-1}$$

as indexes are always at least 1 (Proposition 1.8.3(1)). Note $0 \le \mathrm{tr}_B(p_i) \le 1$ and $\sum_{i=1}^k \mathrm{tr}_B(p_i) = 1$. This sum is minimised when $\mathrm{tr}_B(p_i) = k^{-1}$ for each i. Hence,

$$[B : A] \ge k^2 \tag{3.1}$$

whenever there exists a partition of unity of size k in $A' \cap B$.

If $A' \cap B$ is infinite-dimensional, a partition of unity can always be refined into a larger one, contradicting (3.1) as $[B : A] < \infty$. Hence $A' \cap B$ is finite dimensional and so isomorphic to $\bigoplus_{j=1}^{m} M_{n_j}(\mathbb{C})$ for some n_j, m. By taking matrices with 1 in one diagonal entry and 0 elsewhere, we get a partition of unity $\{p_i\}$ of size $k = \sum_{j=1}^{m} n_j$. It follows that

$$[B : A] \geq \left(\sum_{j=1}^{m} n_j \right)^2 \geq \sum_{j=1}^{m} n_j^2 = \dim(A' \cap B).$$

□

Corollary 3.2.2. *If* $[B : A] < 4$*, then* $A' \cap B = \mathbb{C}1$.

Proof. If $A' \cap B$ were nontrivial, it would contain a partition of unity of size $k > 1$, implying $[B : A] \geq 4$ by (3.1). □

Proposition 3.2.1 shows that each tower $\{M'_m \cap M_n\}_{n \geq m}$ consists of finite-dimensional algebras, so it's a valuable source of invariants. In fact, there exists a 2-shift, a family of trace- and inclusion-preserving isomorphisms $M'_m \cap M_n \to M'_{m+2} \cap M_{n+2}$. (See, e.g. [Bis97, 2.13]; also [PP86] [PP88].) Hence no data is lost by restricting attention to the first two towers (called towers despite being drawn horizontally below).

Definition 3.2.3. Suppose $N \subset M$ is a II_1 subfactor. Then the *standard invariant of* $N \subset M$ consists of the following towers. The *derived tower* of $N \subset M$ refers to the first tower, while the *dual derived tower* refers to the second tower.

$$\begin{array}{ccccccccc} \mathbb{C}1 = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C}1 = M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \dots \end{array}$$

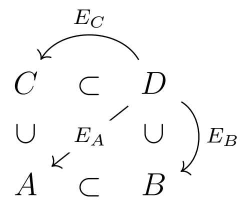
The dual derived tower is so-called because it is itself the derived tower of $M \subset M_1$, which is antilinearly isomorphic by $\operatorname{Ad} J$ to $M' \subset N'$, called the dual subfactor. Hence results about derived towers also apply to the dual, so we never state results specifically for the latter.

It is worth clarifying precisely what data is included in these invariants. The derived tower consists of the algebras $N' \cap M_n$, the inclusion maps $N' \cap M_n \subset N' \cap M_{n+1}$, and the restriction of tr to each algebra. The standard invariant consists of the data of both derived towers *in addition* to the inclusion maps between towers, i.e. $M' \cap M_n \subset N' \cap M_n$.

As its name suggests, the standard invariant has become the most important invariant of a II_1 subfactor. As it consists of finite-dimensional objects, it is far more tractable than the Jones tower itself. We will compute the standard invariant of a II_1 subfactor that we have already seen: the Jones subfactor $J^{(n)} \subset J$, defined in Definition 2.8.5. However, first we must update our algebraic tools.

Commuting squares

The conditional expectation is a strong tool for probing towers of algebras, where the inclusions form a linear sequence. To work with the standard invariant, we need to handle *squares* of inclusions. Suppose A, B, C, D are tracial von Neumann algebras and the following trace-preserving inclusions hold.



The diagram suggests that E_C is a 'horizontal expectation' and E_B is a 'vertical expectation'. But, in general, this is inappropriate terminology. If $x \in B$, then *a priori*, $E_C(x)$ is in C but possibly not A; i.e. the map may 'move diagonally' instead of horizontally. The *commuting square condition* [Pop83a] is the appropriate requirement to make this terminology coherent.

Definition 3.2.4. Suppose A, B, C, D are tracial von Neumann algebras and the above square of tracial inclusions holds. It is a *commuting square* if any of the following four equivalent identities, called commuting square conditions, hold.

$$E_C \circ E_B = E_A \text{ or } E_B \circ E_C = E_A, \tag{3.2}$$

$$E_C|_B = E_A|_B \text{ or } E_B|_C = E_A|_C. \tag{3.3}$$

For a proof of the equivalence, see [GHJ89, 4.2.1] [Pop83b, 2.1].

Proposition 3.2.5. *If $A \subset B$ is an inclusion of tracial von Neumann algebras and $S \subset B$ is a self-adjoint set, the following square commutes:*

$$\begin{array}{ccc} A & \subset & B \\ \cup & & \cup \\ S' \cap A & \subset & S' \cap B \end{array}$$

See [GHJ89, 4.2.7]. In particular, this implies that every square formed by the standard invariant (as shown in Definition 3.2.3) is a commuting square. Equipped with Proposition 3.2.5, we can now compute a standard invariant.

3.3 Standard invariant of the Jones subfactor

We compute the standard invariant of the Jones subfactor $J^{(m)} \subset J$, defined in Definition 2.8.5 for $m \geq 3$. We briefly recall its construction[3]. By Theorem 2.5.11, the type A_{m-1} Coxeter diagram[4] induces a Jones tower with Markov modulus $\tau = 1/\left(4\cos^2(\pi/m)\right)$ when $m \geq 3$. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ denote the sequence of Jones projections in this tower.

Recall $\{\varepsilon_i\}_{i\in\mathbb{N}}$ satisfies the Jones relations with Markov modulus τ, by Theorem 2.7.3. Recall $[i : j]$ is the $*$-algebra generated by $1, \varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_j$. Then $J^{(m)} \subset J$ is defined by $J^{(m)} = [2 : \infty]''$, $J = [1 : \infty]''$, with index $\tau^{-1} = 4\cos^2(\pi/m)$. As $[1 : \infty], [2 : \infty]$ are representations of $TL(\tau)$, the Temperley-Lieb algebra[5], we say $J^{(m)} \subset J$ is a *subfactor generated by Temperley-Lieb*.

Unusually, we will explicitly form the Jones tower, then compute the standard invariant from within it – in general, the Jones tower is extremely hard to describe. By definition of the Jones tower, $J_n = \langle J_{n-1}, e_n\rangle$. Expanding J_{n-1}, J_{n-2}, and so on in this way, it's clear J_n is generated by $J \cup \{e_1, e_2, \ldots, e_n\}$. J is by definition generated by $\{\varepsilon_i\}_{i=1}^{\infty}$. From Corollary 2.8.4, recall that, if we write $\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}, \ldots$ as notation for $e_1, e_2, e_3, \ldots$, then the extended family $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ also satisfies the Jones relations. Therefore, we can write $J_n = [1 - n : \infty]''$.

We now compute the standard invariant. Actually, it suffices to compute the derived tower; the same procedure works for the dual tower.

Proposition 3.3.1. *Suppose $J^{(m)} \subset J$ is the Jones subfactor for $m \geq 3$. Then, for $n \geq -1$,*

$$(J^{(m)})' \cap J_n = [1 - n : 0]. \tag{3.4}$$

To gain intuition for this proposition, observe that, *by definition*,

$$(J^{(m)})' \cap J_n = [2 : \infty]' \cap [1 - n : \infty]'' \tag{3.5}$$

By the Jones relations, if ε_j commutes with $[2 : \infty]$, then $j \leq 0$. Hence, the only $\varepsilon_j \in [2 : \infty]' \cap [1 - n : \infty]''$ are $\varepsilon_{1-n}, \ldots, \varepsilon_0$. Therefore, proving (3.4) amounts to

[3] Actually, just one possible construction, as we explained above Definition 2.8.5.

[4] Also known as the linear graph on $m - 1$ vertices.

[5] By Proposition 2.7.5.

showing that $(J^{(m)})' \cap J_n$ *is generated by the* ε_j *it contains.* A one-line summary is in [GHJ89, 4.7.b.]. We expand it to a proof.

Proof. We prove the $n = -1, 0$ cases separately. In both cases, $[1-n:0]$ is the trivial algebra $\mathbb{C}1$. As J_{-1} is a factor, $(J_{-1})' \cap J_{-1} = \mathbb{C}1$. As $[P : J^{(m)}] = \tau^{-1} < 4$, Corollary 3.2.2 implies $(J^{(m)})' \cap J = \mathbb{C}1$. This finishes the $n = -1, 0$ cases.

For the general case, fix $n \geq 1$. As $[1-n:0]$ and $[2:\infty]$ commute, the inclusion $[1-n:0] \subset (J^{(m)})' \cap J_n$ is immediate. For the reverse inclusion of (3.4), we write $F_i : J_n \to [i:\infty]' \cap J_n$ for the trace-preserving conditional expectation for $i \geq 2$. The inclusion we need to prove is equivalent to $\operatorname{im} F_2 \subset [1-n:0]$.

We first prove a similar inclusion, 'shifted up by one': $\operatorname{im} F_3 \subset [1-n:1]$. A necessary condition is that $\varepsilon_i \notin \operatorname{im} F_3$ for $i \geq 2$. We will show that $F_3(\varepsilon_i)$ is a scalar. In fact, we can show the stronger[6] fact that $F_{i+1}(\varepsilon_i)$ is a scalar for $i \geq 2$.

By Proposition 3.2.5, the following square is commuting.

$$\begin{tikzcd} {[i+1:\infty]' \cap [1-n:\infty]''} & \subset & {[1-n:\infty]''} \arrow[ll, bend right, "F_{i+1}"'] \arrow[dl, "\mathrm{tr}"] \\ \cup & & \cup \\ {[i+1:\infty]' \cap [i:\infty]''} & \subset & {[i:\infty]'';} \end{tikzcd}$$

The top horizontal conditional expectation is F_{i+1}, by definition.

By a shift of indices, the bottom-left corner $[i+1:\infty]' \cap [i:\infty]''$ is isomorphic to $[2:\infty]' \cap [1:\infty]'' = (J^{(m)})' \cap J = \mathbb{C}1$, where the first equality is by (3.5), and the latter was shown at the start of this proof. Thus the diagonal expectation is an expectation onto a trivial subalgebra, which is just the trace.

By the commuting square condition, diagonal expectation agrees with horizontal expectation when applied to the bottom-right corner, i.e. $\mathrm{tr}\,|_{[i:\infty]''} = F_{i+1}|_{[i:\infty]''}$. As $\varepsilon_i \in [i:\infty]''$,

$$F_{i+1}(\varepsilon_i) = \mathrm{tr}(\varepsilon_i) \tag{3.6}$$

as we aimed to prove. As mentioned, we will leverage this to show $\operatorname{im} F_3 \subset [1-n:1]$. As $\operatorname{im} F_3 = F_3(J_n) = F_3\left([1-n:\infty]''\right)$, to show $\operatorname{im} F_3 \subset [1-n:1]$ it suffices to prove $F_3([1-n:k]) \subset [1-n:1]$ for all $k \geq 1$.

Induct on k. This is immediate when $k = 1$ as $[1-n:1] \subset \operatorname{im} F_3$ so F_3 acts as the identity on it. For $k \geq 2$, assume the $k-1$ case holds. Let w be a word in

[6] Stronger as $F_3 \circ F_{i+1} = F_3$ for $i \geq 2$, by the tower property (Proposition 2.2.8).

$\varepsilon_{1-n}, \ldots, \varepsilon_k$. By Proposition 2.7.8, up to scaling $w = u\varepsilon_k v$ where u, v are words in $\varepsilon_{1-n}, \ldots, \varepsilon_{k-1}$. Then,

$$\begin{aligned} F_3(u\varepsilon_k v) &= F_3(F_{k+1}(u\varepsilon_k v)) && (F_3 \circ F_{k+1} = F_3 \text{ for } k \geq 2) \\ &= F_3(uF_{k+1}(\varepsilon_k)v) && (u, v \in [k+1 : \infty]' \cap J_n) \\ &= \operatorname{tr}(\varepsilon_k) F_3(uv) && (\text{by } 3.6) \\ &\in F_3([1-n : k-1]) \\ &\subset [1-n : 1] \end{aligned}$$

by induction, and this proves $\operatorname{im} F_3 \subset [1-n : 1]$.

To show this implies $\operatorname{im} F_2 \subset [1-n : 0]$, take $x \in \operatorname{im} F_2 = [2 : \infty]' \cap J_n$. Certainly $x \in \operatorname{im} F_3$, so $x \in [1-n : 1]$. Hence $x \in [2 : \infty]' \cap [1-n : 1]$.

A priori, x is a limit of linear combinations of words in $\varepsilon_{1-n}, \ldots, \varepsilon_1$. Notably, $\varepsilon_1 \notin [2 : \infty]'$. We'd like to leverage this fact to show ε_1 can be excluded. Without loss of generality, we can assume x is a limit of reduced words, i.e.

$$x = \lim_{l \to \infty} u_l \varepsilon_1 v_l$$

where u_m, v_m are words in $\varepsilon_{1-n}, \ldots, \varepsilon_0$. Let E be the conditional expectation onto $[2 : \infty]' \cap [1-n : 1]$. Applying E,

$$x = \lim_{l \to \infty} u_l E(\varepsilon_1) v_l$$

We verify that $E(\varepsilon_1) = \tau$, using the characterisation of E by Proposition 2.2.7. For any $y \in [2 : \infty]' \cap [1-n : 1]$,

$$\operatorname{tr}(\varepsilon_1 y) = \tau^{-1} \operatorname{tr}(\varepsilon_1 y \varepsilon_2) = \tau^{-1} \operatorname{tr}(\varepsilon_2 \varepsilon_1 \varepsilon_2 y) = \operatorname{tr}(\varepsilon_2 y) = \tau \operatorname{tr}(y).$$

The first and last equality hold by the Markov property (Definition 2.7.1); the second holds because ε_2 commutes with y; the third holds by the second algebraic Jones relation. Hence, $E(\varepsilon_1) = \tau$. It follows that $x = \lim_{l \to \infty} \tau u_l v_l$, and so $x \in [1-n : 0]$, finishing the proof. □

The consequence of Theorem 3.3.1 is that the derived tower $\{J' \cap J_n\}_{n \geq 0}$ of $J^{(m)} \subset J$ is identical to the sequence $\{[1-n : 0]\}_{n \geq -1}$. Applying the proposition *mutatis mutandis* to the dual subfactor $J \subset J_1$ implies that the dual derived tower $\{(J^{(m)})' \cap J_n\}$ is identical to $\{[1-n : -1]\}_{n \geq 0}$. Together, these towers form the standard invariant of $J^{(m)} \subset J$.

In fact, by applying the trace-preserving reflection, we can see the following:

Corollary 3.3.2. *The standard invariant of $J^{(m)} \subset J$ is isomorphic to the pair $\{[2 : n]\}_{n\geq 0}, \{[1 : n]\}_{n\geq -1}$ by an inclusion- and trace-preserving isomorphism sending $e_i \mapsto \varepsilon_i$.*

Proof. Reflect $[1-n : 0]$ and $[1-n : -1]$ onto $[1 : n]$ and $[2 : n]$, respectively, with the trace-preserving reflection sending $\varepsilon_{1-i} \mapsto \varepsilon_i$. But recall (as discussed above Theorem 3.3.1) ε_{1-i} is just notation for e_i, so indeed the isomorphism maps e_i to ε_i. □

Because the isomorphism is inclusion-preserving, it extends to an isomorphism on closures of unions. That is, $\overline{\bigcup_{n\geq 0} J' \cap J_n} \subset \overline{\bigcup_{n\geq -1} (J^{(m)})' \cap J_n}$ is isomorphic to $\overline{\bigcup_{n\geq 0}[2 : n]} \subset \overline{\bigcup_{n\geq -1}[1 : n]}$.

But clearly $\overline{\bigcup_{n\geq 0}[2 : n]} = \overline{[2 : \infty]} = [2 : \infty]'' = J^{(m)}$, and by the same argument we have that $\overline{\bigcup_{n\geq 0}[1 : n]} = J$. That is, $\overline{\bigcup_{n\geq 0} J' \cap J_n} \subset \overline{\bigcup_{n\geq -1} (J^{(m)})' \cap J_n}$ is *isomorphic to* $J^{(m)} \subset J$. In other words, $J^{(m)} \subset J$ can be *recovered from its standard invariant by taking a union and a closure.*

More generally, certain families of well-behaved subfactors can always be reconstructed from their standard invariants, although via more complicated constructions than simply taking a union and a closure [Ocn88, p134] [Pop90, p33] [Pop94]. Owing to this, subfactor theorists today typically work on classifying standard invariants rather than II_1 subfactors themselves [JMS14] [Izu+15] [AMP15]. As standard invariants consist of finite-dimensional objects, this is far simpler than classifying general II_1 subfactors, but still very challenging.

In this challenge, the *principal graphs* Γ, Γ' are an essential tool. They are combinatorial invariants of intermediate strength between the standard invariant and the index. They contain enough data that, with (much) work, it is possible to (nonuniquely[7]) recover a standard invariant from them.

3.4 The full Bratteli diagram

Fix a finite index subfactor $N \subset M$. A key goal of this chapter is to construct the principal graphs Γ, Γ' for $N \subset M$. To do this, we will first extract a larger graph β, the *full Bratteli diagram*, from the standard invariant, and later (in Section 3.2) reduce β to Γ.

To form β, we discard enough structure from the standard invariant so that the remaining data is combinatorial. By Definition 3.2.3, the data of the standard

[7] See Table 3.2 for examples of principal graphs which come from more than one standard invariant.

invariant consists of the derived tower $\{N' \cap M_i\}_{i=-1}^{\infty}$, dual tower $\{M' \cap M_i\}_{i=0}^{\infty}$, inclusion maps between towers, and a trace on each algebra.

We discard the traces, as they are non-combinatorial. Although inclusion maps are specified by inclusion matrices and hence combinatorial, we discard the inclusions *between* towers. The advantage is that there is no loss of generality to restrict attention to the derived tower[8]. This means we will only construct Γ; one can construct Γ' by applying the results of Sections 3.4-3.5 to the dual tower.

Write $Y_i := N' \cap M_i$. The remaining data consists of the algebras Y_i and the inclusion maps *within* the tower, i.e. $Y_i \subset Y_{i+1}$. Henceforth, when we refer to the tower $\{Y_i\}_{i\geq -1}$, we assume that it only contains this data.

This data can be described by graphs. Recall from the discussion following Definition 1.5.10 that the (non-tracial) algebraic data of a finite-dimensional inclusion $A \subset B$ is *completely* encoded by a pair $(\vec{n}^A, \Lambda_A^B)$ or the pair $(\vec{n}^A, \beta_A^B)$, where Λ_A^B is an inclusion matrix and β_A^B is a Bratteli diagram.

In fact, the entire tower can be encoded in a graph. Let the inclusion matrix of $Y_i \subset Y_{i+1}$ be Λ_i^{i+1} and its Bratteli diagram be β_i^{i+1}. Let P_i be the set of minimal central projections (MCPs) of Y_i, where Y_i has the semisimple decomposition $Y_i = \bigoplus_{p\in P_i} Y_i p$ by Theorem 1.5.5. By definition, β_i^{i+1} has left vertices P_i and right vertices P_{i+1}, and Λ_i^{i+1} has row indices P_i and column indices P_{i+1}.

Compose the β_i^{i+1} to form the *full Bratteli diagram* β. As $Y_{-1} = N' \cap N = \mathbb{C}1$ is trivial, P_{-1} contains only one element; denote the corresponding vertex by $*$. See Figure 3.1 for an example.

Definition 3.4.1. The *full Bratteli diagram* of the derived tower $\{Y_i\} = \{N' \cap M_i\}$ is the graph[9] β on the vertex set $\bigsqcup_{i=-1}^{\infty} P_i$ that is formed as the union of the graphs β_i^{i+1} for $i \geq -1$, with distinguished vertex $*$.

Proposition 3.4.2. *The tower $\{Y_i\}_{i\geq -1}$ (including the inclusion maps $Y_i \subset Y_{i+1}$) is uniquely determined by β.*

Proof. By the remarks given above, the data of $Y_i \subset Y_{i+1}$ is fully encoded by the graphs $\{\beta_i^{i+1}\}_{i\geq -1}$ and the initial dimension vector $\vec{n}^{Y_{-1}}$. This initial vector is always (1), so it suffices to show that the subgraphs β_i^{i+1} can be recovered from β. This amounts to showing that the partition of $V(\beta)$ into $\{P_i\}_{i\geq -1}$ can be recovered, which is the result of Lemma 3.4.3. □

[8]The principal graphs Γ, Γ' can be equipped with extra structure to retain some data about the relationship between towers, but we do not consider this.

[9]As usual, meaning an undirected multigraph with unlabelled vertices.

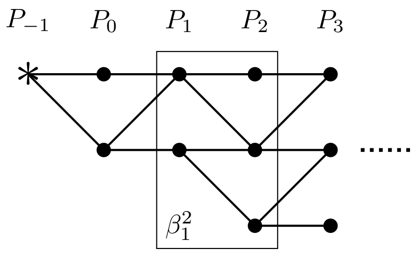


Figure 3.1: The first few levels of a possible β.

As we consider the vertices of β to be unlabelled, the partition $V(\beta) = \bigsqcup_{i=-1}^{\infty} P_i$ is not strictly included in the data of β, but it can be recovered.

Lemma 3.4.3. *$P_i \subset V(\beta)$ is the set of vertices of distance $i+1$ from $*$.*

Proof. P_i only connects to adjacent levels $P_{i\pm1}$, and each sub-Bratteli diagram β_i^{i+1} has no isolated vertices due to Lemma 1.5.12. □

We say that the set P_i is the ith 'level' of β, while the subgraph Γ_{i-1}^{i} is the ith 'storey'[10].

Proposition 3.4.2 shows that β encodes all data of $\{Y_i\}_{i\geq -1}$. We can similarly define β' for the dual derived tower. Then the pair (β, β') encodes the data of the standard invariant, excluding the traces and inclusions between towers. This is a major loss of data, but has the advantage of being represented by a graph.

Unfortunately, β is by definition an infinite graph. Fortunately, β contains numerous redundancies. By removing these, we will form the principal graph Γ, which contains equivalent data to β but can be finite.

Partial basic constructions in the derived tower

To give an analogy for what is meant by 'redundancy', examine the Jones tower $\{M_i\}_{i\geq -1}$ at the level of triplets:

$$M_{i-1} \subset M_i \subset M_{i+1}, \text{ with distinguished projection } e_{i+1}.$$

By definition, M_{i+1} is the basic construction of $M_{i-1} \subset M_i$. This means the tower is highly redundant, as we can recover the tower from its lowest two levels by taking the basic construction of $N \subset M$, then $M \subset M_1$, and so on.

[10]Hence the lowest storey is the 0th, much like a building that houses a computer science department.

The derived tower $\{Y_i\}_{i\geq -1} = \{N' \cap M_i\}_{i\geq -1}$ is *not* a Jones tower; it is 'subtower' of a Jones tower. Nonetheless, it can also be arranged into triplets with distinguished projections.

$$Y_{i-1} \subset Y_i \subset Y_{i+1}, \text{ with distinguished projection } e_{i+1}. \tag{3.7}$$

We would like to test if the derived tower has a similar kind of redundancy to the Jones tower. To do, so we apply the basic construction to $Y_{i-1} \subset Y_i$, and produce the following triplet:

$$Y_{i-1} \subset Y_i \subset \langle Y_i, e_{Y_{i-1}}\rangle, \text{ with distinguished projection } e_{Y_{i-1}}. \tag{3.8}$$

We want to know if the basic construction triplet (3.8) is isomorphic to the triplet (3.7) (meaning an isomorphism that preserves inclusions and distinguished projections). In general, the answer is 'no'. The key reason is that Y_{i+1} is too large to be the basic construction of $Y_{i-1} \subset Y_i$. It contains nontrivial ideals which are mutually orthogonal with e_{i+1}. This is data that is 'independent' from e_{i+1}, and cannot come from a basic construction.

However, we can remedy this by cutting out the offending ideal.

Definition 3.4.4. The central support of a projection p in a von Neumann algebra A is the smallest projection $z_A(p)$ in $Z(A)$ such that $p \leq z_A(p)$, i.e.

$$z_A(p) := \bigwedge \{q \in Z(A) : p \leq q\}$$

Definition 3.4.5. For $i = -1, 0$, let $z_i = 0$ and $X_i = 0$. For $i \geq 1$, let $z_i = z_{Y_i}(e_i)$ denote the central support of e_i in Y_i, and let $X_i := Y_i z_i$ be the ideal generated by z_i in Y_i.

The ideal $Y_i(1 - z_i)$ is the largest ideal in Y_i which is 'independent' from e_i. To eliminate this ideal, we cut down the triplet (3.7) by z_i, to obtain the triplet $Y_i z_{i+2} \subset Y_{i+1} z_{i+2} \subset X_{i+2}$. Remarkably, the cut-down triplet *is* isomorphic to the basic construction triplet (3.8).

Theorem 3.4.6. *Let $i \geq -1$. The following triplets are isomorphic.*

$$[Y_{i-1} z_{i+1} \subset Y_i z_{i+1} \subset^{e_{i+1}} X_{i+1}] \cong \left[Y_{i-1} \subset Y_i \subset^{e_{Y_{i-1}}} \langle Y_i, e_{Y_{i-1}}\rangle\right] \tag{3.9}$$

The isomorphism preserves inclusions and distinguished projections. The restriction of the isomorphism to $Y_i z_{i+1}$ is the inverse of $x \mapsto x z_{i+1}$.

A proof of this nontrivial fact is found in [Pop90, 2.1]. This is a remarkable result which means that the derived tower is 'partially redundant'.

To see this, decompose Y_{i+1} as $Y_{i+1} = X_{i+1} \oplus Y_{i+1}(1 - z_{i+1})$. Theorem 3.4.6 means X_{i+1} is isomorphic to the basic construction of $Y_{i-1} \subset Y_i$. I.e., while Y_{i+1} is *not* the basic construction of $Y_{i-1} \subset Y_i$, it *contains* an isomorphic copy of it.

In particular, this means that the structure of X_{i+1} can be completely inferred from the previous two levels of the tower, Y_{i-1} and Y_i. Conversely, nothing about the complementary ideal $Y_{i+1}(1 - z_{i+1})$ is determined by lower levels. We call X_{i+1} the 'old ideal' or 'old stuff' of Y_{i+1}, whereas $Y_{i+1}(1 - z_{i+1})$ is 'new'.

Recall that β fully encodes $\{Y_i\}_{i\geq -1}$. We will construct Γ from β by discarding 'old stuff', and prove that no data is lost, i.e. Γ is equivalent to β. We then obtain an invariant that also encodes $\{Y_i\}_{i\geq -1}$, but improves on β by possibly being finite.

3.5 The principal graph

The partition of Y_i into 'old' and 'new' $Y_i = X_i \oplus Y_i(1 - z_i)$, induces a partition of the vertices of β into 'old and new'.

To show this, recall that the ith level of β is the set P_i of minimal central projections of Y_i. This set decomposes as $P_i = \{p \in P_i : p \in X_i\} \sqcup \{p \in P_i : p \in Y_i(1 - z_i)\}$. By Theorem 3.4.6, X_i is isomorphic to the basic construction of $Y_{i-2} \subset Y_{i-1}$, so Lemma 2.5.2 implies the set of MCPs of X_i is the set $\tilde{P}_i = \{\tilde{p} : p \in P_{i-2}\}$, where $\tilde{p} \in Z(X_i)$ is unique such that $pe_i = \tilde{p}e_i$.

Write P_i^{new} for the set of MCPs of the new ideal $Y_i(1 - z_i)$, so we can write the partition as $P_i = \tilde{P}_i \sqcup P_i^{\text{new}}$. Vertices in $\tilde{P}_i$ are called old vertices, and vertices in P_i^{new} are called new vertices. We summarise this below: P_i^{new} and $\tilde{P}_i$ have two and three equivalent definitions, respectively:

$$\tilde{P}_i = \{p \in P_i : p \in X_i\} = \{p \in P_i : pz_i = p\} = \{\tilde{p} : p \in P_{i-2}\} \tag{3.10}$$

$$P_i^{\text{new}} = \{p \in P_i : p \in Y_i(1 - z_i)\} = \{p \in P_i : pz_i = 0\} \tag{3.11}$$

In β, draw each old vertex $\tilde{p}$ so it is the reflection of p 'across' the $(i-1)$th level, as in Figure 3.2. The principal graph Γ is constructed from β by discarding the old vertices. See Figure 3.3 for a comparison of β and Γ.

Definition 3.5.1. The principal graph Γ of a subfactor $N \subset M$ is the induced subgraph of β on vertices $\bigsqcup_{i\geq -1} P_i^{\text{new}}$, with distinguished vertex $*$.

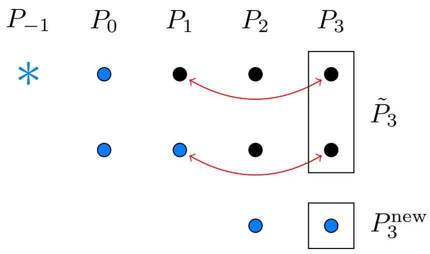
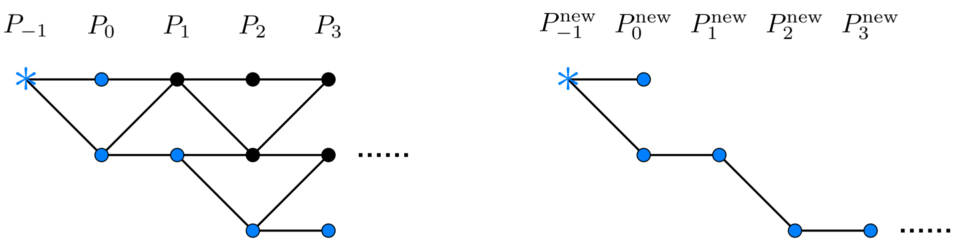


Figure 3.2: Some vertices of β; edges omitted. Black vertices are old and blue vertices are new.



(a) Full Bratteli diagram β.

(b) Principal graph Γ.

Figure 3.3: β and its associated Γ.

In fact, *no data is lost* in passing from β to Γ.

Theorem 3.5.2. *If two II_1 subfactors have the same principal graph Γ, then they have the same full Bratteli diagram β.*

(The proof concludes after Theorem 3.5.5.) The task at hand is to show that vertices and edges not belonging to Γ represent *redundant data*. We write that an edge is 'old-old' if it has old left endpoint and old right endpoint. Similarly, we write 'old-new', etc. As Γ is induced by new vertices, it contains exactly the new-new edges. We'll show that the three other types of edge are redundant data.

First, we show that new-old and old-old edges are redundant. Let $\tilde{\beta}_{i-1}^i$ be the induced subgraph of β_{i-1}^i on $P_{i-1} \sqcup \tilde{P}_i$. That is, $\tilde{\beta}_{i-1}^i$ contains exactly the edges of β_{i-1}^i whose right endpoint is old – namely, new-old and old-old edges. See Figure 3.4; observe that the right endpoints of $\tilde{\beta}_{i-1}^i$ are black (old).

Observe in Figure 3.4 that β_i^{i+1} is a mirror image of the preceding storey. Proposition 3.5.3 states that this is a general rule.

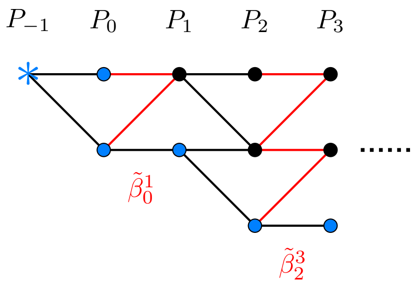


Figure 3.4: β from Figure 3.3 with $\tilde{\beta}_0^1$ and $\tilde{\beta}_2^3$ highlighted.

Proposition 3.5.3. *For $i \geq 1$, $\tilde{\beta}_{i-1}^i$ is the reflection of β_{i-2}^{i-1} across P_{i-1}.*

Proof. The left vertices of $\tilde{\beta}_{i-1}^i$ are P_{i-1}, i.e. the MCPs of Y_{i-1}. By Theorem 3.4.6, $Y_{i-1} \cong Y_{i-1}z_i$, so P_{i-1} is also identified with the set of MCPs of $Y_{i-1}z_i$. The right vertices of $\tilde{\beta}_{i-1}^i$ are $\tilde{P}_i$, which are the MCPs of $X_i = Y_i z_i$, by (3.10).

Because $\tilde{\beta}_{i-1}^i$ is an induced subgraph of a Bratteli diagram, it follows that $\tilde{\beta}_{i-1}^i$ is itself the Bratteli diagram of the inclusion $Y_{i-1}z_i \subset Y_i z_i$.[11]

It follows that $\tilde{\beta}_{i-1}^i$ has biadjacency matrix $\Lambda_{Y_{i-1}z_i}^{Y_i z_i} = \Lambda_{Y_{i-1}z_i}^{X_i}$. To show that $\tilde{\beta}_{i-1}^i$ is the reflection of β_{i-2}^{i-1}, it suffices to show that their biadjacency matrices are transposes of one another. The latter has biadjacency matrix $\Lambda_{Y_{i-2}}^{Y_{i-1}}$, by the definition made above Definition 3.4.1.

By Theorem 3.4.6, $Y_{i-2}z_i \subset Y_{i-1}z_i \subset X_i$ is a basic construction triplet. Then, by Lemma 2.5.3,

$$\Lambda_{Y_{i-1}z_i}^{X_i} = (\Lambda_{Y_{i-2}z_i}^{Y_{i-1}z_i})^T = (\Lambda_{Y_{i-2}}^{Y_{i-1}})^T$$

where the second equality holds because $x \mapsto xz_i$ is an isomorphism of $Y_{i-2} \subset Y_{i-1}$ onto $Y_{i-2}z_i \subset Y_{i-1}z_i$ (Theorem 3.4.6). □

Therefore, the subgraph $\tilde{\beta}_{i-1}^i$ is redundant data, because it is determined by the previous (i.e. the $(i-1)$th) storey of β. As remarked above Proposition 3.5.3, $\tilde{\beta}_{i-1}^i$ consists exactly of new-old and old-old edges. Hence, edges of these types are redundant. In Figure 3.5, we draw all redundant edges as dashed lines, and non-redundant (meaningful) edges as solid.

[11]We could show this with more care, but it is a cumbersome finite-dimensional algebraic argument that provides little intuition.

We want the principal graph Γ to encode all meaningful information, but there is an issue: Γ, by definition, consists of new-new edges, but the meaningful edges may include old-new edges! See the red edge in Figure 3.5.

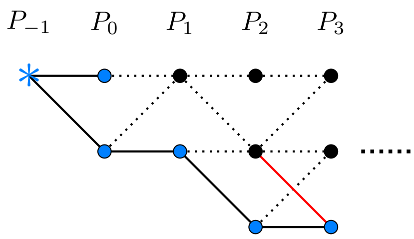


Figure 3.5: β, with redundant edges shown as dashed lines.

In fact, this is no problem at all.

Proposition 3.5.4. *There are no old-new edges in β.*

Proof. There are no old vertices in the (-1)st and 0th levels, so it suffices to prove that there are no edges from $\tilde{P}_{i-1}$ to P_i^{new} when $i \geq 2$.

Let $\tilde{p} \in \tilde{P}_{i-1}$ and $q \in P_i^{\text{new}}$. By definition of Bratteli diagrams (Definition 1.5.10, the number of edges from $\tilde{p}$ to q represents the number of copies of $Y_{i-1}\tilde{p}$ that are 'embedded inside' $Y_i q$. To show this is zero, it suffices to show that $\tilde{p}q = 0$.

By (3.10), (3.11), $\tilde{p} \leq z_{i-1}$ and $q \leq 1 - z_i$. So it clearly suffices to show $z_{i-1}(1 - z_i) = 0$. As $z_{i-1} \in X_{i-1}$, we'll prove the more general fact that $X_{i-1}(1 - z_i) = 0$.

Because X_{i-1} is the basic construction of $Y_{i-3} \subset Y_{i-2}$, Lemma 2.2.11 implies that $Y_{i-2} + \operatorname{span} Y_{i-2}e_{i-1}Y_{i-2}$ is dense in X_{i-1}. It is a nontrivial fact [Pop90, 2.1] [PP86, 1.3] that this can be strengthened: $\operatorname{span} Y_{i-2}e_{i-1}Y_{i-2}$ is dense in X_{i-1}.

Since $(1 - z_i) \in Z(Y_i)$, it commutes with Y_{i-2}, so it suffices to show $e_{i-1}(1 - z_i) = 0$. Using the Jones relations (Theorem 2.6.4),

$$\begin{aligned} e_{i-1}(1 - z_i) &= [M : N]e_{i-1}e_i e_{i-1}(1 - z_i) \\ &= [M : N]e_{i-1}e_i(1 - z_i)e_{i-1} \end{aligned} \tag{3.12}$$

As z_i is the central support of e_i (Definition 3.4.5), in particular $e_i \leq z_i$ and so $e_i z_i = e_i$. Hence, we conclude from (3.12) that $e_{i-1}(1 - z_i) = 0$. In particular, this proves that $\tilde{p}q = 0$, and so there are no edges from $\tilde{P}_{i-1}$ to P_i^{new}. □

Proposition 3.5.4 implies that edges in β^i_{i-1} are either old-old, new-old, or new-new. By definition, the former two types are edges of $\tilde{\beta}^i_{i-1}$. By definition, the latter are edges of Γ^i_{i-1}. The following is then immediate.

Theorem 3.5.5. *The ith storey*[12] *of β is the edge-disjoint union of a reflection of the $(i-1)$th storey of β, and the ith storey of Γ.*

Observe in Figure 3.6 that each storey has a dashed part which reflects the previous storey, and a blue part.

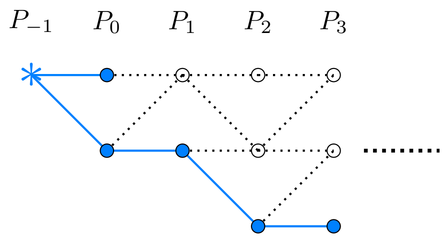


Figure 3.6: Blue edges/vertices are in Γ. Dashed edges are in β but not Γ.

We can therefore recover β from Γ by successively reflecting each storey, as shown in Figure 3.7. Hence Γ is a faithful invariant of β, proving Theorem 3.5.2. This means that Γ fully encodes the combinatorial (i.e., non-tracial) data of the tower $\{Y_n\}_{n\geq -1} = \{N' \cap M_n\}_{n \geq -1}$. Unlike β, Γ is the 'minimal' encoding, as all redundancies of β are removed. This is extremely important: while β is never finite, Γ can be finite, unlocking far more graph-theoretic techniques.

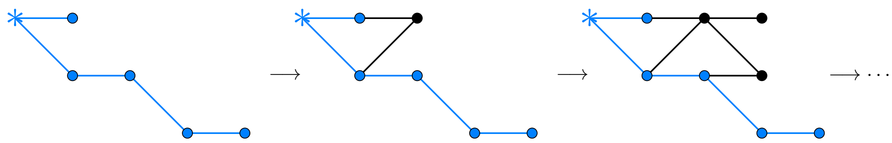


Figure 3.7: Reconstructing β from Γ.

Also, this makes Γ a good indicator of tractability for a subfactor. A II_1 subfactor $N \subset M$ with finite principal graphs is extremely well-behaved, because its derived towers truly contain a finite amount of information. Therefore, we are very interested in a notion of size for Γ.

[12] Recall that the ith storey is the subgraph between the $(i-1)$th level and the ith level.

3.6 The depth of a principal graph

A natural notion of size for Γ is its *depth*. We describe a method for computing it, and hence determining when Γ is finite. We then prove one of the most important results about finite principal graphs: that the index $[M : N]$ can be recovered from the principal graph of $N \subset M$, as long as the graph is finite. This will conclude our long quest to find an invariant that generalises the index.

Definition 3.6.1. depth(Γ) is, equivalently, the maximal distance of any vertex from $*$ in Γ, or the smallest $d \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ such that the dth level of Γ is unoccupied.

Recall that the ith level of Γ is P_i^{new}. The two definitions are equivalent by the following lemma, which is similar to an earlier result for β (Lemma 3.4.3).

Lemma 3.6.2. *$P_i^{\text{new}} \subset V(\Gamma)$ is the set of vertices of distance $i+1$ from $*$.*

Proof. Let $p \in P_i^{\text{new}}$. Since Γ is a subgraph of β and $P_i^{\text{new}} \subset P_i$, Lemma 3.4.3 implies that there exists a path from $*$ to p in β of length $i+1$. Since $*$ and p are new vertices, either every edge in this path is new-new, in which case it is a path in Γ, or else the path contains at least one old-new edge. By Proposition 3.5.4, such edges don't exist in β. Therefore, $d(p, *) = i + 1$ in Γ. □

Computing depth(Γ) is clearly equivalent to the task of determining if an arbitrary level of Γ is unoccupied, so we introduce a technique for the latter. Recall that Λ_{i-1}^i is the biadjacency matrix of β_{i-1}^i, the Bratteli diagram of $Y_{i-1} \subset Y_i$.

Theorem 3.6.3. *(Termination condition for Γ)*
If $i \geq 0$, the $(i+1)$th level of Γ is unoccupied if and only if $\|\Lambda_{i-1}^i\|^2 = [M : N]$.

Theorem 3.6.3 is a miracle – *a priori*, there is no clear reason to suggest that Γ should be related to the index $[M : N]$ at all! The key link between Γ and the index is given by the following lemma. (The proof of Theorem 3.6.3 appears after the proof of this lemma.)

Lemma 3.6.4. *$\|\Lambda_{i-1}^i\|^2 \leq [M : N]$ for all $i \geq 0$.*

Proof. This proof is adapted from [GHJ89, 4.6.3(v)]. Suppose to the contrary that there exists $\epsilon > 0$ and $i \geq 0$ such that $\|\Lambda_{i-1}^i\| > (1-\epsilon)^{-1}[M : N]$. We will derive a contradiction of Lemma 3.2.1, which states $\dim Y_j = \dim(N' \cap M_j) \leq [M : N]$.

Let $\vec{n}^{(j)}$ be the dimension vector of Y_j. By definition, for each entry n_p^j of $\vec{n}^{(j)}$, Y_j has a $M_{n_p^j}(\mathbb{C})$ summand, so its dimension is given by $\dim Y_j = \|\vec{n}^{(j)}\|^2$.

Let $p \geq 0$, and we look $2p$ storeys above the $(i-1)$th level. By Lemma 1.5.11, $\vec{n}^{(j)} = \vec{n}^{(j-1)}\Lambda_{j-1}^j$. Repeatedly applying this identity,

$$\dim Y_{i-1+2p} = \|\vec{n}^{(i-1+2p)}\|^2 = \|\vec{n}^{(i-1)}\Lambda_{i-1}^i \Lambda_i^{i+1} \cdots \Lambda_{i-2+2p}^{i-1+2p}\|^2 \tag{3.13}$$

We determine what the inclusion matrices Λ_j^{j+1} look like. By Theorem 3.5.5, β_j^{j+1} is an edge-disjoint union of $\tilde{\beta}_j^{j+1}$ and Γ_j^{j+1}. Because the subgraphs $\tilde{\beta}_j^{j+1}$ and Γ_j^{j+1} are bipartite and their right-vertex sets are disjoint, the biadjacency matrix Λ_j^{j+1} of β_j^{j+1} must be (up to permutation) formed by horizontally concatenating the subgraphs' biadjacency matrices. $\tilde{\beta}_j^{j+1}$ is the reflection of β_{i-1}^i, so it has biadjacency matrix $(\Lambda_{i-1}^i)^T$. Let Ξ be the biadjacency matrix of Γ_j^{j+1}. Hence,

$$\Lambda_j^{j+1} = \begin{pmatrix} (\Lambda_{j-1}^j)^T & \Xi \end{pmatrix}. \tag{3.14}$$

We can apply the same identity to Λ_{j-1}^j, then Λ_{j-2}^j, and so on. Hence, whenever $j \geq i$, Λ_j^{j+1} contains Λ_{i-1}^i as a submatrix (a corner) if $j-i$ is odd, and otherwise it contains $(\Lambda_{i-1}^i)^T$ as a corner.

Since inclusion matrices have nonnegative entries, replacing every matrix in (3.13) with a corner does not cause the norm to increase. There are $2p$ matrices in (3.13), so we get

$$\dim Y_{i-1+2p} \geq \|\vec{n}^{(i-1)}\left(\Lambda_{i-1}^i(\Lambda_{i-1}^i)^T\right)^p\|^2 = \|\left(\Lambda_{i-1}^i(\Lambda_{i-1}^i)^T\right)^p\vec{n}^{(i-1)}\|^2 \tag{3.15}$$

Write Λ for Λ_{i-1}^i. Because Λ is an inclusion matrix, the reasoning in the proof of Theorem 2.5.11 shows that $\Lambda\Lambda^T$ is the adjacency matrix of some nonempty connected graph. Hence the Perron-Frobenius theorem (Theorem 2.5.10) implies $\Lambda\Lambda^T$ has an eigenvector $\vec{\nu}$ with strictly positive entries and eigenvalue $\|\Lambda\|^2$. Scale $\vec{\nu}$ so it is entry-wise smaller than $\vec{n}^{(i)}$. Then, by comparing to (3.15),

$$\dim Y_{i-1+2p} \geq \|\left(\Lambda\Lambda^T\right)^p\vec{\nu}\|^2 = \|\Lambda\|^{4p}\|\nu\|^2$$
$$\frac{(\dim Y_{i-1+2p})^{1/(i+2p)}}{\|\Lambda\|^2} \geq \left(\frac{\|\nu\|^2}{\|\Lambda\|^{2i}}\right)^{1/(i+2p)}$$

The right side converges to 1 as $p \to \infty$. For some $k = i-1+2p$ sufficiently large, we obtain $\dim Y_{k-1} \geq (1-\epsilon)^k\|\Lambda\|^{2k} > [M:N]^k$, where the second inequality follows by the assumption we made at the start.

But by definition, $Y_{k-1} = N' \cap M_{k-1}$, and Lemma 3.2.1 bounds its dimension by $\dim Y_{k-1} \leq [M_k : N] = [M_k : M_{k-1}]\cdots[M_1 : M][M : N] = [M:N]^k$, so we have a contradiction. □

Lemma 3.6.4 is remarkable! *A priori*, the index is a non-discrete measure of size, yet it controls the size and complexity of a combinatorial object. In light of this lemma, Theorem 3.6.3 simply states that Γ terminates when $\|\Lambda_{i-1}^i\|^2$ has reached its maximal value. We can now prove this.

Proof. (Theorem 3.6.3)

$\Longleftarrow$: Suppose, to gain a contradiction, that $\|\Lambda_{i-1}^i\|^2 = [M : N]$ but that level $i+1$ of Γ is occupied, i.e. $\tilde{P}_{i+1}^{\text{new}} \neq \emptyset$. Because each level must be connected to $*$ (Lemma 3.6.2), the level below any occupied level is occupied, so $\tilde{P}_i^{\text{new}} \neq \emptyset$. Hence, Γ_i^{i+1} is a nonempty subgraph of Γ.

From (3.14), if Ξ denotes the biadjacency matrix of Γ_i^{i+1}, then, up to permutation, $\Lambda_i^{i+1} = \Big((\Lambda_{i-1}^i)^T \quad \Xi\Big)$. As Γ_i^{i+1} is nonempty, Ξ is a nontrivial matrix of nonnegative integers, so it must be that $\left\|\Lambda_i^{i+1}\right\|^2 > \left\|\Lambda_{i-1}^i\right\|^2 = [M : N]$, contradicting Lemma 3.6.4.

$\Longrightarrow$: Suppose the $(i+1)$th level of Γ is unoccupied, i.e. $P_{i+1}^{\text{new}} = \emptyset$. By definition (see (3.11)), $P_{i+1}^{\text{new}} = \{p \in P_{i+1} : p \leq (1 - z_{i+1})\}$. This means $1 - z_{i+1}$ has no minimal subprojections, which implies $1 - z_{i+1} = 0$.

By Theorem 3.4.6, $z_{i+1} = 1$ implies $X_{i+1} = Y_{i+1}$, and hence $Y_{i-1} \subset Y_i \subset^{e_{i+1}} Y_{i+1}$ is isomorphic to a basic construction triplet. As $Y_{i+1} \subset M_{i+1}$, where M_{i+1} is a II_1 factor, we can restrict the trace of M_{i+1} to Y_{i+1}. Because the Jones tower $\{M_i\}_{i \geq -1}$ has the Markov property with modulus $[M : N]^{-1}$, $\text{tr}(xe_{i+1}) = [M : N]^{-1}\,\text{tr}(x)$ for all $x \in M_i$, in particular all $x \in Y_i$. Hence, the triplet $Y_{i-1} \subset Y_i \subset^{e_{i+1}} Y_{i+1}$ itself satisfies the Markov relation with modulus $[M : N]^{-1}$.

But recall from Proposition 2.5.9 that the Frobenius property (Definition 2.5.7) is a good tower-building property – in particular, a recoverably tower-building property (Definition 2.3.6). Then, because $Y_{i-1} \subset Y_i \subset^{e_{i+1}} Y_{i+1}$ is Markov, it follows that $Y_{i-1} \subset Y_i$ is Frobenius with parameter $[M : N]$. That is, if $\vec{\tau}$ is the trace vector of $\text{tr}\,|_{Y_{i-1}}$, then

$$(\Lambda_{i-1}^i)^T \Lambda_{i-1}^i \vec{\tau} = [M : N]\vec{\tau}.$$

Hence $\left\|(\Lambda_{i-1}^i)\right\|^2 \geq [M : N]$. By Lemma (3.6.4), the reverse inequality holds, so we conclude that $\left\|(\Lambda_{i-1}^i)\right\|^2 = [M : N]$ whenever level $i+1$ is unoccupied. This proves Theorem 3.6.3. □

Theorem 3.6.3 provides a method for computing depth(Γ), as long as it is finite: one computes the sequence $\{\Lambda_{i-1}^{Y_i}\}_{i \geq 0}$ until it stabilises to a 2-periodic sequence $(\Lambda_{d-1}^d), (\Lambda_{d-1}^d)^T, \ldots$ Then one simply reads off the level where stabilisation occurs. This is summarised below:

Corollary 3.6.5. $\mathrm{depth}(\Gamma) = 1 + \min\{i \geq 0 : \|\Lambda^i_{i-1}\|^2 = [M : N]\}$.

When $\mathrm{depth}(\Gamma) < \infty$, much more can be said about Γ. For example, because the 2-periodic sequence of biadjacency matrices alternates between a matrix and its transpose, the sequence of Bratteli diagrams $\{\beta^i_{i-1}\}$ stabilises to a constant sequence (as a matrix and its transpose represent the same bipartite graph up to isomorphism). It is a remarkable fact that it stabilises *to the principal graph.*

This means, despite the fact that Γ is *a priori* defined as a subgraph of the infinite graph Γ, we can compute it by calculating only a finite sequence of graphs (as long as Γ is actually finite).

Theorem 3.6.6. *If* $\mathrm{depth}(\Gamma) = d < \infty$*, then the Bratteli diagram of* $Y_{d-1} \subset Y_d$ *is isomorphic to* Γ*.*

Proof. Write $G = \beta^d_{d-1}$ for the Bratteli diagram of $Y_{d-1} \subset Y_d$.

We define an equivalence relation $\sim$ on $\bigsqcup_{i=1}^d P_i$, generated by the equivalence[13] of $p \in P_{i-2}$ with $\tilde{p} \in \tilde{P}_i \subset P_i$ for all $1 \leq i \leq d$. Then, it's clear that each equivalence class C consists of a single new vertex in addition to old vertices at successively higher levels (in steps of two), i.e. $C = \{p_j, p_{j+2}, p_{j+4}, \ldots, p_k\}$, where $k \in \{d, d-1\}$, $p_j \in P^{\mathrm{new}}_j$, and $p_i \in \tilde{P}_i$ for $j + 2 \leq i \leq k$.

$V(G) = P_{d-1} \sqcup P_d$, so we define $\varphi : V(G) \to V(\Gamma)$ by mapping $p \in V(G)$ to the unique new vertex in its equivalence class. This is easily seen to be bijective as each equivalence class contains only one vertex in $P_{d-1} \sqcup P_d$ and every new vertex in $V(\Gamma)$ appears in some equivalence class. To see that this map is edge-preserving, suppose p_{d-1}, q_d are a pair of vertices in P_{d-1}, P_d connected by at least one edge (the case where they are disconnected is straightforward). Suppose $\varphi(p_{d-1})$ and $\varphi(q_d)$ are at the ith and jth levels, respectively, assuming $i < j$, and we write $\varphi(p_{d-1}) = p_i$ and $\varphi(q_d) = q_j$.

Then, write the vertices in the equivalence classes of p and q in ascending order of level as follows: $p_i, p_{i+2}, \ldots, p_{j-1}, q_j, p_{j+1}, q_{j+2}, \ldots, p_{d-1}, q_d$.

Each of these is old besides p_i and q_j. An old-old edge is a reflection of an edge at a lower storey by Proposition 3.5.3, so the number of edges from p_{j-1} to q_j is the same as between q_j and p_{j+1}, and so on. In particular, there must be at least one edge from p_{j-1} to q_j. Since old-new edges don't exist by Proposition 3.5.4, p_{j-1} must be new, so $j - 1 = i$ since each equivalence class has exactly one new representative. Then the edge number between $\varphi(p_{d-1}) = p_{j-1}$ and $\varphi(q_d) = q_j$ is the same as between p_{d-1} and q_d, so φ is an edge-preserving bijection. □

[13] See (3.10).

Assuming Γ is finite, Theorem 3.6.6 provides a way to compute Γ in finite time: construct the sequence $\{\beta^i_{i-1}\}_{i\geq 0}$, and stop when it stabilises. In practice, one can often use subtler graph-theoretic arguments to compute Γ, which we do in Section 3.8. Arguably, the most important outcome of Theorem 3.6.6 is not the algorithm, but the following:

Corollary 3.6.7. *If* $\operatorname{depth}(\Gamma) < \infty$*, then* $\|\Gamma\|^2 = [M : N]$.

Proof. Let $d = \operatorname{depth}(\Gamma)$. The graph norm $\|\Gamma\|$ is just the norm of its adjacency matrix. By Theorem 3.6.6, $\Gamma \cong \beta^d_{d-1}$, which has *bi*adjacency matrix Λ^d_{d-1}, but this has the same norm as the adjacency matrix. By the termination condition (Theorem 3.6.3), $\|\Lambda^d_{d-1}\|^2 = [M : N]$, and it follows that $\|\Gamma\|^2 = [M : N]$. □

Corollary 3.6.7 is immensely important. It shows, in the finite-depth case, the principal graph generalises the index. Therefore, the three major invariants of II_1 subfactor theory fit neatly into an ordering by strength:

$$\text{Standard invariant} \;\geq\; \text{Principal graphs} \;\geq\; \text{Index}$$

The unifying idea of these three invariants is that they represent different amounts of information about the Jones tower. The standard invariant stores only the Jones tower's finite-dimensional data, the principal stores only finite-dimensional *combinatorial* data, and the index stores only the Markov modulus.

3.7 The principal graph as a finer invariant

The principal graph is a vital object, as it allows the considerable power of combinatorics and graph theory to apply to subfactor theory. Consider Kronecker's theorem [GHJ89, 1.1.1] which states that, if Λ is a nonzero matrix of integers, then $\|\Lambda\|^2 \in \{4\cos^2(\pi/n) : n = 3, 4, 5, \ldots\} \cup [4, \infty)$. This should look familiar! We use this fact to give a very short alternative proof of half of the index theorem.

Theorem 3.7.1. *(Jones index theorem - constraint on the index)*
If $N \subset M$ *is a* II_1 *subfactor,* $[M : N] \in \{4\cos^2(\pi/n) : n = 3, 4, 5, \ldots\} \cup [4, \infty]$.

Proof. Assume $[N : M] < 4$. We'll show that $N \subset M$ is a finite depth subfactor. Supposing $N \subset M$ has infinite depth, there are vertices in Γ at level $i + 1$ for all $i \geq 0$. By an argument near the start of the proof of Theorem 3.6.3,

$$4 > [M : N] \geq \left\|\Lambda^{i+1}_i\right\|^2 > \left\|\Lambda^i_{i-1}\right\|^2 \text{ for all } i \geq 0. \tag{3.16}$$

But, by Kronecker's theorem, $\|\Lambda_i^{i-1}\|^2 \in \{4\cos^2(\pi/n) : n = 3, 4, 5, \ldots\}$. This set has a single limit point at 4, so the increasing sequence of (3.16) can't exist.

Therefore, by contradiction, $N \subset M$ has finite depth and hence Γ is a finite graph. But Γ is a connected graph with $\|\Gamma\|^2 = [M : N] \leq 4$, and so again by Kronecker's theorem $[M : N] \in \{4\cos^2(\pi/n) : n = 3, 4, 5, \ldots\}$. □

This proof appears in [GHJ89, 4.6.6] six years after Jones's original proof [Jon83, 4.3.1]. They share an approach: from the Jones tower, one extracts an invariant which strictly generalises the index (a Temperley-Lieb algebra in [Jon83] and a principal graph in [GHJ89]). The constraint on the index follows from a constraint on the stronger object. However, the principal graph lets us say more.

Corollary 3.7.2. *(Constraint on principal graphs)*

If $N \subset M$ is a II_1 subfactor with index $[M : N] < 4$, then its principal graph Γ is a Coxeter diagram of type A, D, or E.

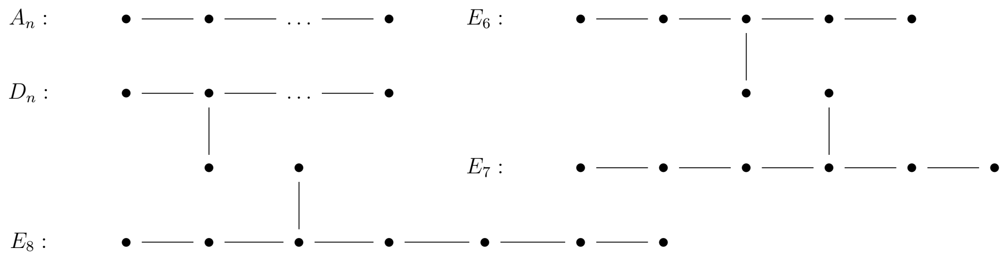


Figure 3.8: The A, D, E Coxeter diagrams. (Diagram by Isabel Longbottom.)

Proof. By the argument from the proof above, Γ has finite depth and so by Corollary 3.6.7, we have $\|\Gamma\|^2 = [M : N] < 4$. It is known that the nonempty finite connected graphs of norm-squared strictly less than 4 are Coxeter diagrams of type A, D, or E. (See Table 3.1.) □

Coxeter diagram	A_n	D_n	E_6	E_7	E_8
Norm-squared	$4\cos^2\left(\frac{\pi}{n+1}\right)$	$4\cos^2\left(\frac{\pi}{2n-2}\right)$	$4\cos^2\left(\frac{\pi}{12}\right)$	$4\cos^2\left(\frac{\pi}{18}\right)$	$4\cos^2\left(\frac{\pi}{30}\right)$

Table 3.1: The squared norms of A, D, E Coxeter diagrams [GHJ89, 1.4.3].

The earliest references where the result of Corollary 3.7.2 appears include [Ocn88, p139] and [GHJ89, 4.6.6]. Efforts by various authors over the following years identified which of the A, D, E Coxeter diagrams are realised by subfactors, and how many distinct standard invariants give rise to each graph.

Coxeter diagram	A_n	D_{2n+1}	D_{2n}	E_6	E_7	E_8
Number of standard invariants	1	0	1	2	0	2

Table 3.2: Number of subfactor standard invariants realising the Coxeter diagrams of type A, D, and E [JMS14] [Ocn88] [GHJ89] [Izu91] [Kaw95].

Theorem 3.6.6 states that a finite-depth principal graph recovers the index, but Table 3.2 shows by example that the principal graph is *strictly* finer than the index – e.g. a subfactor with A_{11} principal graph and a subfactor with E_6 principal graph both have index $4\cos^2(\pi/12) = 2 + \sqrt{3}$.

To bring this thesis to a denouement, we will verify part of Table 3.2: we will show that the Jones subfactor $J^{(n)} \subset J$ has type A principal graph.

3.8 Principal graph of the Jones subfactor

Theorem 3.8.1. *If $n \geq 3$, the principal graph of $J^{(n)} \subset J$ is A_{n-1}.*

This result appears in [GHJ89, 4.7.b].

Proof. Let the principal graph be Γ. Since $[J : J^{(n)}] = 4\cos^2(\pi/n) < 4$, Corollary 3.7.2 constrains Γ to be type A, D, or E. To rule out D and E it suffices to show that Γ has no branching, or equivalently that there is at most one vertex in each level of Γ. By definition (see (3.11)) this is equivalent to the ideal $Y_i(1 - z_i)$ containing at most one minimal central projection for all $i \geq -1$.

By Theorem 3.3.1, the derived tower $\{Y_i\}_{i\geq-1} = \{(J^{(n)})' \cap J_i\}_{i\geq-1}$ is isomorphic to $\{[1 : i]\}_{i\geq-1}$ by a map sending $e_i \mapsto \varepsilon_i$, where the ε_i satisfy Jones relations. Hence we identify Y_i with $[1 : i]$ and z_i with the the central support of ε_i in Y_i. (Recall Definition 3.4.5.) We claim that

$$z_i = \bigvee_{j=1}^{i} \varepsilon_j = \varepsilon_1 \vee \varepsilon_2 \vee \ldots \vee \varepsilon_i. \tag{3.17}$$

As $\bigvee_{j=1}^{i} \varepsilon_i$ commutes with $\varepsilon_1, \ldots, \varepsilon_i$, it is central in $Y_i = [1 : i]$. By Definition 3.4.4, to prove equality, we must show that $\bigvee_{j=1}^{i} \varepsilon_i$ is minimal among central projections that dominate ε_i.

If $z' \in Z(Y_i)$ is another central projection such that $\varepsilon_i \leq z'$, then $z'\varepsilon_i = \varepsilon_i = \varepsilon_i z'$. To show $\bigvee_{j=1}^{i} \varepsilon_i \leq z'$, it suffices to show $\varepsilon_j \leq z'$ for $j = 1, \ldots, i$. The $j = i$ case is true by assumption. We induct on j, so suppose $\varepsilon_{j+1} \leq z'$ (and so

$z'\varepsilon_{j+1} = \varepsilon_{j+1}$). Using the Jones relations,

$$z'\varepsilon_j = \tau^{-1}z'(\varepsilon_j\varepsilon_{j+1}\varepsilon_j) = \tau^{-1}\varepsilon_j(z'\varepsilon_{j+1})\varepsilon_j = \tau^{-1}(\varepsilon_j\varepsilon_{j+1}\varepsilon_j) = \varepsilon_j.$$

$z'\varepsilon_j = \varepsilon_j$ implies $\varepsilon_j \leq z'$, so by induction $\bigvee_{j=1}^{i} \varepsilon_i \leq z'$, proving (3.17).

Armed with this result, we can return to our initial aim: we will show, if $1 - z_i \neq 0$, then $Y_i(1 - z_i)$ has exactly one minimal central projection. We can show something stronger: that the only projections in $Y_i(1 - z_i)$ are 0 and $1 - z_i$. Suppose $p \in Y_i(1 - z_i) \subset Y_i$ is a projection. Then $p \leq 1 - z_i$ and so $p(1 - z_i) = p$. As $Y_i = [1 : i]$, p is a linear combination of words in $\varepsilon_1, \ldots, \varepsilon_i$.

By (3.17), $\varepsilon_j(1-z_i) = 0$ for $j = 1, \ldots, i$. Hence, right-multiplying p by $(1-z_i)$ eliminates every term in p except for the empty word 1. If 1 doesn't appear in p, then $p = p(1 - z_i) = 0$. If 1 does appear, then $p = p(1 - z_i) = 1 - z_i$.

It follows that $Y_i(1 - z_i)$ has either one or zero nontrivial projections. In particular, this means that the ith level of Γ, namely P_i^{new} (see (3.11)), contains one or zero vertices. Hence Γ cannot branch.

It follows from Corollary 3.7.2 that Γ is type A. By Corollary 3.6.7, $\|\Gamma\|^2 = [J : J^{(n)}] = 4\cos^2(\pi/n)$. The type A Coxeter diagram with this norm-squared value is A_{n-1}, by Table 3.1 [GHJ89, 1.4.3]. □

We now know all three major invariants for the Jones subfactor $J^{(n)} \subset J$.

Standard invariant	Principal graph	Index
$\{[2 : n]\}_{n\geq 0}, \{[1 : n]\}_{n\geq -1}$	A_{n-1}	$4\cos^2(\pi/n)$

Table 3.3: The three major invariants of $J^{(n)} \subset J$.

This shows that the Jones subfactors $J^{(n)} \subset J$ are the simplest possible subfactors. They occupy the smallest allowable index values, and their principal graphs are also the simplest possible, as they are linear.

This also reveals the limitations in their construction. To obtain the other principal graphs in Table 3.2 of types other than A, one must resort to more sophisticated constructions [JMS14].

Today, the subfactor classification effort has progressed beyond index 4; to our knowledge, standard invariants are currently classified up to index $5 + 1/4$ [AMP15] [PT12]. Although the techniques presently used are far more sophisticated than those originally introduced by Jones and described in this thesis, their genesis can still be traced, ultimately, back to one groundbreaking paper: 'Index for Subfactors' [Jon83].

Bibliography

[AMP15] Narjess Afzaly, Scott Morrison, and David Penneys. *The classification of subfactors with index at most* $5\frac{1}{4}$. 2015. arXiv: `1509.00038 [math.OA]`.

[Arv76] William Arveson. *An invitation to* C^**-algebras*. Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg, 1976, pp. x+106.

[Bis97] Dietmar Bisch. "Bimodules, higher relative commutants and the fusion algebra associated to a subfactor". In: *Operator algebras and their applications (Waterloo, ON, 1994/1995)*. Vol. 13. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 1997, pp. 13–63. DOI: `10.1007/s002220050137`. URL: `https://doi-org.virtual.anu.edu.au/10.1007/s002220050137`.

[Bla06] B Blackadar. *Operator algebras*. Vol. 122. Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2006, pp. xx+517. ISBN: 978-3-540-28486-4. DOI: `10.1007/3-540-28517-2`. URL: `https://doi-org.virtual.anu.edu.au/10.1007/3-540-28517-2%20https://packpages.unr.edu/media/1224/cycr.pdf`.

[Con00] John B Conway. *A course in operator theory*. Vol. 21. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2000, pp. xvi+372. ISBN: 0-8218-2065-6. DOI: `10.1090/gsm/021`. URL: `https://doi-org.virtual.anu.edu.au/10.1090/gsm/021`.

[Gan59] F R Gantmacher. *The theory of matrices. Vols. 1, 2*. Chelsea Publishing Co., New York, 1959, Vol. 1, x+374 pp. Vol. 2, ix+276.

[GHJ89] Frederick M Goodman, Pierre de la Harpe, and Vaughan F R Jones. *Coxeter graphs and towers of algebras*. Vol. 14. Mathematical Sciences Research Institute Publications. Springer-Verlag, New York, 1989, pp. x+288. ISBN: 0-387-96979-9. DOI: `10.1007/978-1-4613-9641-3`.

URL: https://doi-org.virtual.anu.edu.au/10.1007/978-1-4613-9641-3.

[Izu91] Masaki Izumi. "Application of fusion rules to classification of subfactors". In: *Publ. Res. Inst. Math. Sci.* 27.6 (1991), pp. 953–994. ISSN: 0034-5318. DOI: 10.2977/prims/1195169007. URL: https://doi-org.virtual.anu.edu.au/10.2977/prims/1195169007.

[Izu+15] Masaki Izumi et al. "Subfactors of index exactly 5". In: *Bull. Lond. Math. Soc.* 47.2 (2015), pp. 257–269. ISSN: 0024-6093. DOI: 10.1112/blms/bdu113. URL: https://doi-org.virtual.anu.edu.au/10.1112/blms/bdu113.

[JS97] V Jones and V S Sunder. *Introduction to subfactors.* Vol. 234. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1997, pp. xii+162. ISBN: 0-521-58420-5. DOI: 10.1017/CBO9780511566219. URL: https://doi-org.virtual.anu.edu.au/10.1017/CBO9780511566219.

[Jon83] Vaughan F R Jones. "Index for subfactors". In: *Invent. Math.* 72.1 (1983), pp. 1–25. ISSN: 0020-9910. DOI: 10.1007/BF01389127. URL: https://doi-org.virtual.anu.edu.au/10.1007/BF01389127.

[Jon85] Vaughan F R Jones. "A polynomial invariant for knots via von Neumann algebras". In: *Bull. Amer. Math. Soc. (N.S.)* 12.1 (1985), pp. 103–111. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-1985-15304-2. URL: https://doi-org.virtual.anu.edu.au/10.1090/S0273-0979-1985-15304-2.

[Jon87] Vaughan F R Jones. "Hecke algebra representations of braid groups and link polynomials". In: *Ann. of Math. (2)* 126.2 (1987), pp. 335–388. ISSN: 0003-486X. DOI: 10.2307/1971403. URL: https://doi-org.virtual.anu.edu.au/10.2307/1971403.

[Jon09] Vaughan F R Jones. *Von Neumann Algebras.* 2009. URL: https://math.berkeley.edu/~vfr/MATH20909/VonNeumann2009.pdf.

[Jon14] Vaughan F R Jones. *The Jones polynomial for Dummies.* 2014. URL: https://math.berkeley.edu/~vfr/jonesakl.pdf.

[JMS14] Vaughan F R Jones, Scott Morrison, and Noah Snyder. "The classification of subfactors of index at most 5". In: *Bull. Amer. Math. Soc. (N.S.)* 51.2 (2014), pp. 277–327. ISSN: 0273-0979. DOI: 10.1090/S0273-0979-2013-01442-3. URL: https://doi-org.virtual.anu.edu.au/10.1090/S0273-0979-2013-01442-3.

[Kad04] Richard V Kadison. "Non-commutative conditional expectations and their applications". In: *Operator algebras, quantization, and noncommutative geometry.* Vol. 365. Contemp. Math. Amer. Math. Soc., Providence, RI, 2004, pp. 143–179. DOI: 10.1090/conm/365/06703. URL: https://doi-org.virtual.anu.edu.au/10.1090/conm/365/06703.

[Kau90] Louis H Kauffman. "An invariant of regular isotopy". In: *Trans. Amer. Math. Soc.* 318.2 (1990), pp. 417–471. ISSN: 0002-9947. DOI: 10.2307/2001315. URL: https://doi-org.virtual.anu.edu.au/10.2307/2001315.

[Kaw95] Yasuyuki Kawahigashi. "On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors". In: *J. Funct. Anal.* 127.1 (1995), pp. 63–107. ISSN: 0022-1236. DOI: 10.1006/jfan.1995.1003. URL: https://doi-org.virtual.anu.edu.au/10.1006/jfan.1995.1003.

[Mor17] Scott Morrison. "A formula for the Jones-Wenzl projections". In: *Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday.* Vol. 46. Proc. Centre Math. Appl. Austral. Nat. Univ. Austral. Nat. Univ., Canberra, 2017, pp. 367–378.

[MN37] F J Murray and J von Neumann. "On rings of operators. II". In: *Trans. Amer. Math. Soc.* 41.2 (1937), pp. 208–248. ISSN: 0002-9947. DOI: 10.2307/1989620. URL: https://doi-org.virtual.anu.edu.au/10.2307/1989620.

[MN43] F J Murray and J von Neumann. "On rings of operators. IV". In: *Ann. of Math. (2)* 44 (1943), pp. 716–808. ISSN: 0003-486X. DOI: 10.2307/1969107. URL: https://doi-org.virtual.anu.edu.au/10.2307/1969107.

[MV36] F J Murray and J Von Neumann. "On rings of operators". In: *Ann. of Math. (2)* 37.1 (1936), pp. 116–229. ISSN: 0003-486X. DOI: 10.2307/1968693. URL: https://doi-org.virtual.anu.edu.au/10.2307/1968693.

[Neu40] J v. Neumann. "On rings of operators. III". In: *Ann. of Math. (2)* 41 (1940), pp. 94–161. ISSN: 0003-486X. DOI: 10.2307/1968823. URL: https://doi-org.virtual.anu.edu.au/10.2307/1968823.

[Ocn88] Adrian Ocneanu. "Quantized groups, string algebras and Galois theory for algebras". In: *Operator algebras and applications, Vol. 2.* Vol. 136. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1988, pp. 119–172.

[Pen14] David Penneys. *Introduction to subfactors.* 2014. URL: https://people.math.osu.edu/penneys.2/talks/PenneysNCGOA2014.pdf.

[PT12] David Penneys and James E Tener. "Subfactors of index less than 5, Part 4: Vines". In: *Internat. J. Math.* 23.3 (2012), pp. 18, 1250017. ISSN: 0129-167X. DOI: 10.1142/S0129167X11007641. URL: https://doi-org.virtual.anu.edu.au/10.1142/S0129167X11007641.

[PP86] Mihai Pimsner and Sorin Popa. "Entropy and index for subfactors". In: *Ann. Sci. École Norm. Sup. (4)* 19.1 (1986), pp. 57–106. ISSN: 0012-9593. URL: http://www.numdam.org.virtual.anu.edu.au/item?id=ASENS_1986_4_19_1_57_0.

[PP88] Mihai Pimsner and Sorin Popa. "Iterating the basic construction". In: *Trans. Amer. Math. Soc.* 310.1 (1988), pp. 127–133. ISSN: 0002-9947. DOI: 10.2307/2001113. URL: https://doi-org.virtual.anu.edu.au/10.2307/2001113.

[Pop83a] Sorin Popa. "Maximal injective subalgebras in factors associated with free groups". In: *Adv. in Math.* 50.1 (1983), pp. 27–48. ISSN: 0001-8708. DOI: 10.1016/0001-8708(83)90033-6. URL: https://doi-org.virtual.anu.edu.au/10.1016/0001-8708(83)90033-6.

[Pop83b] Sorin Popa. "Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras". In: *J. Operator Theory* 9.2 (1983), pp. 253–268. ISSN: 0379-4024.

[Pop90] Sorin Popa. "Classification of subfactors: the reduction to commuting squares". In: *Invent. Math.* 101.1 (1990), pp. 19–43. ISSN: 0020-9910. DOI: 10.1007/BF01231494. URL: https://doi-org.virtual.anu.edu.au/10.1007/BF01231494.

[Pop94] Sorin Popa. "Classification of amenable subfactors of type II". In: *Acta Math.* 172.2 (1994), pp. 163–255. ISSN: 0001-5962. DOI: 10.1007/BF02392646. URL: https://doi-org.virtual.anu.edu.au/10.1007/BF02392646.

[Pop95] Sorin Popa. "An axiomatization of the lattice of higher relative commutants of a subfactor". In: *Invent. Math.* 120.3 (1995), pp. 427–445. ISSN: 0020-9910. DOI: 10.1007/BF01241137. URL: https://doi-org.virtual.anu.edu.au/10.1007/BF01241137.

[RS80] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I.* Second. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1980, pp. xv+400. ISBN: 0-12-585050-6.

[Sak56] Shôichirô Sakai. "A characterization of W^*-algebras". In: *Pacific J. Math.* 6 (1956), pp. 763–773. ISSN: 0030-8730. URL: http://projecteuclid.org.virtual.anu.edu.au/euclid.pjm/1103043801.

[Tak02] M Takesaki. *Theory of operator algebras. I.* Vol. 124. Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2002, pp. xx+415. ISBN: 3-540-42248-X.

[TL71] H N V Temperley and E H Lieb. "Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem". In: *Proc. Roy. Soc. London Ser. A* 322.1549 (1971), pp. 251–280. ISSN: 0962-8444. DOI: 10.1098/rspa.1971.0067. URL: https://doi-org.virtual.anu.edu.au/10.1098/rspa.1971.0067.

[Ume54] Hisaharu Umegaki. "Conditional expectation in an operator algebra". In: *Tohoku Math. J. (2)* 6 (1954), pp. 177–181. ISSN: 0040-8735. DOI: 10.2748/tmj/1178245177. URL: https://doi-org.virtual.anu.edu.au/10.2748/tmj/1178245177.

[Wen87] Hans Wenzl. "On sequences of projections". In: *C. R. Math. Rep. Acad. Sci. Canada* 9.1 (1987), pp. 5–9. ISSN: 0706-1994.