Towards a unified framework for the mathematics of conformal field theory

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Part 0: Overview

Axiomatizing CFT

- A (two-dimensional, chiral) conformal field theory comes with a lot of data: correlation functions, fusion products, conformal blocks, braiding, characters, mapping class group representations, tensor categories, central charge, and more.
- The goal is to provide axioms for a *subset* of the data in such a way that:
  1) it is possible to recover the remaining data
  2) expected behavior can be rigorously derived
  3) all physically relevant models satisfy the axioms
- Even in low dimensions this is very difficult, but it also has a knack for producing interesting and broadly applicable mathematics.

Conformal nets vs VOAs

- Conformal nets and vertex operator algebras are two approaches to axiomatizing 2d chiral CFT.
- They axiomatize different subsets of the data, using different mathematical tools.
- When we compare the two descriptions we can physical ideas to relate different areas of mathematics. E.g.

  Conjugate sectors ⇝ (CNs) Finite Jones index

  Conjugate sectors ⇝ (VOAs) Invertible fusion matrix

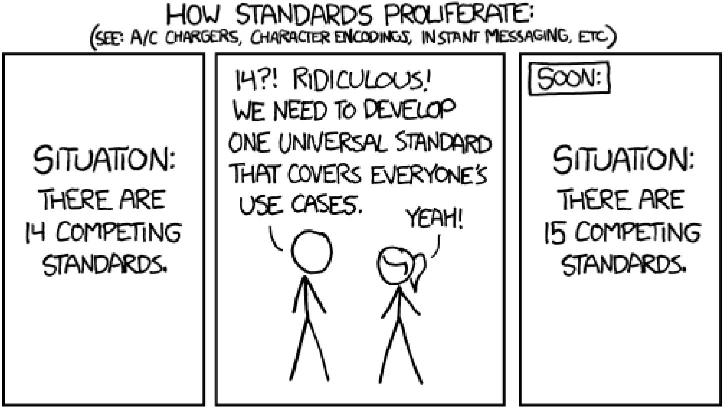
- Note: these are axiomatizations of *unitary* CFTs.

Multiple axiomatizations

Once you have axioms, what sorts of problems might you study?

1. Translate a physical statement into the appropriate language and try to prove it; e.g. "The orbifold of a rational CFT by a finite group is again rational." For CNs and VOAs this has been ongoing for about 30 years.
2. Translate a physical statement into different axiomatizations, and try to prove that the resulting statements are equivalent. For CNs and VOAs this has been rapidly developing for the past 5 to 10 years.
3. Develop an axiomatization which unifies existing approaches and allows algebraic/analytic/geometric tools to be used together. This work is happening now.

We also need to avoid:



(Source: Randall Munroe, https://xkcd.com/927/)

Part 1: Genus zero

Conformal nets

A conformal net $\mathcal{A}$ is

- a Hilbert space $\mathcal{H}_0$ and a unit vector $\Omega \in H_0$
- for every interval $I \subset S^1$, a von Neumann algebra $\mathcal{A}(I) \subset B(H_0)$

◦ a projective unitary representation U of $\mathrm{Diff}_+(S^1)$ on H_0

such that

- if $I \subset J$ then $\mathcal{A}(I) \subset \mathcal{A}(J)$
- if $I \cap J = \emptyset$ then $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute

◦ $U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma(I))$

◦ if $\mathrm{supp}(\gamma) \subset I$ then $U(\gamma) \in \mathcal{A}(I)$

◦ if γ extends holomorphically to the disk $\mathbb{D}$, then $U(\gamma)\Omega = \Omega$

◦ Ω is cyclic for the $\mathcal{A}(I)$

Versions of this definition first appear in Fredenhagen-Rehren-Schroer '92 and Gabbiani-Fröhlich '93, following Haag-Kastler '64.

Representations of conformal nets

A representation of a conformal net is given by:

- a family of representations $\lambda_I : \mathcal{A}(I) \to \mathcal{B}(H_\lambda)$, compatible with inclusion of intervals

From this we extract a subfactor:

$$\underbrace{\lambda_{I'}\big(\mathcal{A}(I')\big)}_{N} \subseteq \underbrace{\lambda_I\big(\mathcal{A}(I)\big)'}_{M}$$

which has a Jones-Kosaki index $[M : N] =: \text{index}(\lambda)$.

Theorem (Jones '83)

The set of possible subfactor indices is

$$\{4\cos^2(\pi/n) \; : \; n = 3, 4, \ldots\} \cup [4, \infty].$$

Question (Jones-Wassermann '90s)

Where do the subfactors of index less than 4 "come from"?

Examples of conformal nets: WZW models

- G - compact simple simply connected Lie group
- LG - the loop group $C^\infty(S^1, G)$
- $\pi_{k,0}$ - the level k vacuum representation of $\widetilde{LG}$ for $k \in \mathbb{Z}_+$

WZW models are given by:

$$\mathcal{A}_{G,k}(I) = \mathrm{vNA}\left(\{\pi_{k,0}(f) \,:\, \mathrm{supp}(f) \subseteq I\}\right)$$

Representations of $\mathcal{A}_{G,k}$ correspond to positive energy representations of LG, i.e. positive energy representations of the affine Lie algebra $\widetilde{L^0\mathfrak{g}}$ (Henriques '19, Gui '21).

Rational conformal nets

Definition

A conformal net is called completely rational if it has finitely many iso classes of irreducible representations, each with finite index.

An apparently stricter definition first appeared in Kawahigashi-Longo-Müger '01, later simplified in Longo-Xu '04 and Morinelli-Tanimoto-Weiner '18.

Theorem (KLM '01 [+ LX '04 + MTW '18])

If $\mathcal{A}$ is a completely rational conformal net, then $\mathrm{Rep}(\mathcal{A})$ *is naturally a unitary modular tensor category.*

The rigidity of $\mathrm{Rep}(\mathcal{A})$ corresponds to finiteness of the index.

Rationality of WZW conformal nets

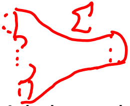
- It is difficult to show that CN representations have finite index (see Gabbiani-Fröhlich '93).
- Wassermann '98 showed that all irreps of type A WZW conformal nets have finite index. Followed by Toledano Laredo '97 in type D, odd level. Field theoretic calculations done 'by hand.'
- Gui '18 used smeared intertwining operators and VOA theory to prove complete rationality of type CG WZW nets.

Theorem (T '19)

All WZW conformal nets are completely rational (i.e. the associated subfactors have finite index). The same holds for discrete series W-algebras of type ADE.

Proof uses new geometric methods for translating rigidity between CNs and VOAs.

Segal CFTs in the vacuum sector



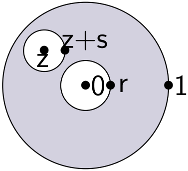
Data of a vacuum Segal CFT:

- A Hilbert space H_0
- For every n-to-1 genus zero Riemann surface Σ with boundary components parametrized by S^1, a map $Y_\Sigma : \bigotimes_{\partial_{in}\Sigma} H_0 \to H_0$

Such that:

- Gluing of surfaces $\longleftrightarrow$ composition of maps (up to scalar)
- Y_Σ is holomorphic in Σ

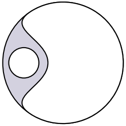
The dictionary between vacuum Segal CFT and VOAs is:



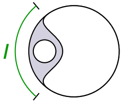
$$= \Sigma \quad \longmapsto \quad Y_\Sigma(v \otimes u) = Y(s^{L_0} v, z) r^{L_0} u$$

Thin surfaces

- Given a vacuum Segal CFT ($\cong$ VOA), we can consider densely defined maps $Y_\Sigma : H_0 \otimes H_0 \to H_0$ for “thin” surfaces like

$\Sigma =$ 

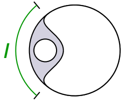
- These maps can be obtained as limits of “thick” surfaces
- A VOA is called integrable if the maps $Y_\Sigma(v \otimes -) : H_0 \to H_0$ are continuous for all Σ as above.
- Theorem [T ’19, Henriques-T]: Given an integrable VOA , the following is a conformal net:

$$\mathcal{A}(I) = vNA\left(\left\{ Y_\Sigma(v \otimes \cdot) \;\middle|\; \Sigma = \right.\right.$$ 

$$\left.\left.\right\}\right)$$

Conformal nets $\leftrightarrow$ VOAs

$$\mathcal{A}(I) = vNA\left(\left\{ Y_\Sigma(v \otimes \cdot) \;\middle|\; \Sigma = \right.\right.$$

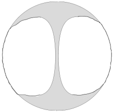


$$\left.\left.\right\}\right)$$

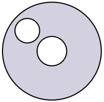
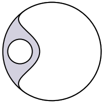
- Most of your favorite unitary VOAs are known to be integrable, and conjecturally all are.
- Deep results (e.g. rigidity) may be passed back and forth via this relationship
- Article in preparation with Henriques shows that *every* conformal net comes from an integrable VOA.
- Combining with joint work with Raymond and Tanimoto, every conformal net is also generated via smeared fields $\oint Y(v,z)f(z)\,dz$.

Extended vacuum Segal CFTs

- An *extended* vacuum Segal CFT also assigns maps partially thin surfaces such as

$\Sigma =$  , $\qquad Y_\Sigma : H_0 \otimes H_0 \to H_0.$

- Conjecture that every vacuum Segal CFT extends to these surfaces, and known that it does for large classes of examples.
- Extended vacuum Segal CFTs contain the data of both VOAs and CNs by respectively specializing to:

 and 

- Showing that a VOA/CN has an extended vacuum Segal CFT gives analytic info about the VOA and algebraic info about the CN.

Part 2: Higher genus

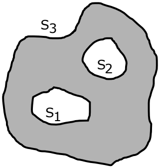
Segal CFT beyond the vacuum

- Chiral CFTs also assign geometric invariants to higher genus surfaces
- These provide examples of what Segal calls "weakly conformal field theories" and I'll call "Segal CFTs."
- Unlike VOAs/conformal nets/vacuum Segal CFTs, the axioms of Segal CFT are not 'minimal.' The data of a Segal CFT includes all of the representations, all of the categorical information, and more.

Example: WZW models

- $\mathfrak{g}$ compact simple Lie algebra, $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g}_\mathbb{C})$.
- We describe Segal's proposed tensor structure on $\mathcal{C} := \mathrm{Rep}_k(L\mathfrak{g})$, the category of positive energy representations of $L\mathfrak{g}$.
- There is a "tensor product" for every complex pair of pants with parametrized boundary.

$\Sigma =$



$\longleftrightarrow \; - \boxtimes_\Sigma - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

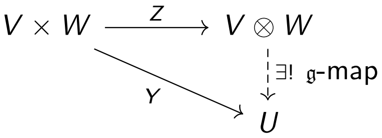
Warmup: tensor product of Lie algebra modules

If V, W are $\mathfrak{g}$-modules, the tensor product $V \otimes W$ is a $\mathfrak{g}$-module:

- equipped with a bilinear map $Z : V \times W \to V \otimes W$, such that

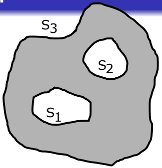
$$X \cdot Z(v, w) = Z(X \cdot v, w) + Z(v, X \cdot w)$$

  for all $X \in \mathfrak{g}$
- which is universal, so that if U is a $\mathfrak{g}$-module equipped with $Y : V \times W \to U$, then Y factors through Z.



Existence of such a module is shown by explicit construction.

Segal's holomorphic induction

$\Sigma =$ 

$\longleftrightarrow - \boxtimes_\Sigma - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

If $V_1, V_2 \in \mathcal{C} = \mathrm{Rep}_k(L\mathfrak{g})$, then $V_1 \boxtimes_\Sigma V_2$ is a $L\mathfrak{g}$-module:

- equipped with a bilinear map $Z_\Sigma : V_1 \times V_2 \to V_1 \boxtimes_\Sigma V_2$ satisfying the *Segal commutation relations:*

$$f|_{S_3} \cdot Z_\Sigma(v, w) = Z_\Sigma(f|_{S_1} \cdot v, w) + Z_\Sigma(v, f|_{S_2} \cdot w)$$

  for all $f \in \mathcal{O}_{hol}(\Sigma; \mathfrak{g}_\mathbb{C})$

- which is universal

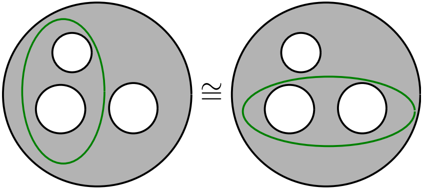
Problems: Existence? Positivity energy? Associativity?

- Note: we can do the same procedure for any Riemann surface Σ

Associativity

The associativity property says that we should have an equivalence between the follows functors

$$\left(\mathcal{C} \times \mathcal{C}\right) \times \mathcal{C} \to \mathcal{C} \cong \mathcal{C} \times \left(\mathcal{C} \times \mathcal{C}\right) \to \mathcal{C}$$



We also want the maps Z_Σ to be compatible with these isomorphisms.

Unitary chiral Segal CFT

1) For every smooth, oriented 1-manifold S,
   a) a Hilb-linear category $\mathcal{C}(S)$,
   b) equipped with a functor $H_\bullet : \mathcal{C}(S) \to \mathrm{Hilb}$.

2) For every complex cobordism Σ with $\partial_{out}\Sigma \neq \emptyset$
   a) a functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$
   b) a natural transformation $Z_\Sigma : H_\lambda \to H_{F_\Sigma(\lambda)}$ $(\lambda \in \mathcal{C}(\partial_{in}\Sigma))$

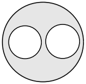
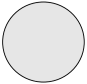
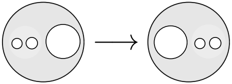
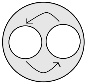
3) For every path of surfaces $[0,1] \ni t \mapsto \Sigma_t$
   a) an equivalence $F_{\Sigma_0} \cong F_{\Sigma_1}$
   b) "chirality", "projectively flat"

Part (a) corresponds to a modular functor, Part (b) is the CFT.

Tensor category structure

Segal CFT answers the question "What is the structure of the representation theory of a chiral CFT?"

The (a) part makes $\mathcal{C}(S^1)$ into a braided tensor category.

- Tensor product $- \boxtimes - \ =$ 
- Unit 1 $=$ 
- Associator $(- \boxtimes -) \boxtimes - \ \cong \ - \boxtimes (- \boxtimes -)$: a path 
- Braiding $- \boxtimes - \ \cong \ (- \boxtimes -) \circ \text{flip}$: a path 

The (b) part can be very interesting even when the (a) part is trivial (e.g. Moonshine CFT). In genus zero, morally equivalent to vertex tensor category.

Examples

- Because the Segal CFT has so much data, it is very difficult to construct examples.
- Segal CFTs in this spirit has been constructed for some lattice models (Posthuma '12) and the free fermion (T '17).
- Ongoing project with Henriques to construct *enhanced* Segal CFTs from an arbitrary *enhanced* genus zero Segal CFT (i.e. from a conformal net with a nice analytic property)
- The resulting structure controls all aspects of the CFT: the conformal net and its representation category, the VOA, a vertex tensor category, mapping class group representations, and so on.

Thank you!