A unitary VOA for every conformal net

James Tener
Australian National University

based on joint work with André Henriques [arXiv:2507.20735]

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Axiomatic quantum field theory

> *"In theoretical physics, since the time of Newton, the axiomatic method has served not only for the systematization of results previously obtained, but also in the discovery of new results."*
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- A QFT comes with a lot of data: fields, correlation functions, observables, OPEs, scattering matrix, etc.

- The goal is to provide axioms for a *subset* of the data in such a way that:
  1) it is possible to recover the remaining data
  2) expected behavior can be rigorously derived
  3) all physical models satisfy the axioms

- This has proven to be very challenging.

Vertex operator algebras (VOAs)

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$$V = \bigoplus_{d=0}^{\infty} V(d).$$

- The key data of a VOA is a 'state-field correspondence':

$$Y(v,z) = \sum_{n \in \mathbb{Z}} v_n \, z^{-n-d}, \qquad v_n : V \to V.$$

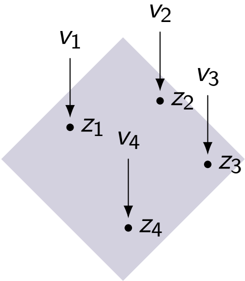
- There is a vacuum vector $\Omega \in V(0)$, and we have $v = \lim_{z \to 0} Y(v,z)\Omega$.
- There is a vector $\nu \in V(2)$ whose modes give a representation of the Virasoro algebra $Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$

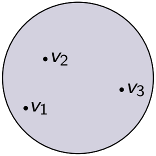
- Given $v_1, \ldots, v_n \in V$ and distinct $z_1, \ldots, z_n \in \mathbb{C}$ with local coordinates, we have a vector

$$|v_1(z_1) \cdots v_n(z_n)\rangle \in \widehat{V} := \prod_{d=0}^{\infty} V(d).$$

- This is holomorphic in the z_j, and when $|z_1| > |z_2| \cdots > |z_n|$

$$|v_1(z_1) \cdots v_n(z_n)\rangle = Y(v_1, z_1) \cdots Y(v_n, z_n)\Omega.$$





- A *unitary* VOA has a compatible inner product, and

$$|z_j| < 1 \quad \implies \quad |v_1(z_1) \cdots v_n(z_n)\rangle \in H := \bigoplus^{\ell^2} V(d).$$

VOAs and the Wightman axioms on the circle

- The key axiom that the fields $Y(v, z)$ satisfy is *locality*.
- Locality is easiest to formulate in terms of smeared fields

$$Y(v, f) = \oint_{S^1} f(z) Y(v, z) \frac{dz}{2\pi\imath z^{1-d}}, \qquad f \in C^\infty(S^1).$$

which are linear maps $Y(v, f) : \mathcal{D} \to \mathcal{D}$ for a certain $\mathcal{D} \supset V$.

- The smeared fields are operator-valued distributions.
- Locality says that $Y(v_1, f_1)$ and $Y(v_2, f_2)$ commute when f_1 and f_2 have disjoint support.

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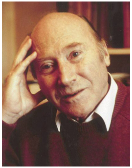
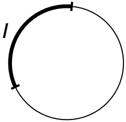
- The smeared fields are operator-valued distributions.
- Locality says that $Y(v_1, f_1)$ and $Y(v_2, f_2)$ commute when f_1 and f_2 have disjoint support.
- The link between VOA axioms and the Wightman axioms has been recognized for a long time.
- In [Carpi-Raymond-Tanimoto-T, '25], we showed that they are exactly equivalent (even for non-unitary theories!).

Algebraic CFT





- The Haag-Kastler axioms for a unitary QFT describe *nets of algebras of observables*. In the context of 2d chiral CFT, these are called *conformal nets*.
- A conformal net assigns (von Neumann) algebras on a Hilbert space H to intervals of S^1:

$$I \subset S^1 \quad \mapsto \quad \mathcal{A}(I) \subset B(H)$$

- Locality says that $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute when $I \cap J = \emptyset$.
- There is also a vacuum vector Ω and a unitary representation of the Virasoro algebra L_n.

The VOA $\leftrightarrow$ conformal net correspondence

- A unitary VOA V should correspond to a conformal net $\mathcal{A}_V$:

$$\mathcal{A}_V(I) = \mathsf{vNA}\left(\left\{ Y(v,f) \,\middle|\, v \in V,\ \mathrm{supp}(f) \subset I \right\}\right).$$

- The expected correspondence was carefully described in [Carpi-Kawahigashi-Longo-Weiner '18]

The Question:

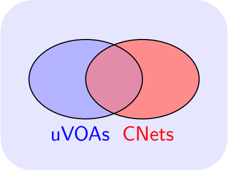
> *"The question as to whether the Wightman axioms are equivalent to a theory formulated in terms of a net of algebras of bounded operators has been the subject of extensive discussions... One may ask under what conditions the construction of von Neumann algebras... leads to a net respecting the causal structure... Conversely one may ask whether, given a net of algebras of bounded operators, one can define fields..."*
>
> — *R. Haag, Local Quantum Physics (1992)*

- Does every unitary VOA produce a conformal net?
- Does every conformal net come from a unitary VOA?

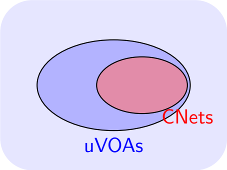
Main Result

Theorem (Henriques-T '25)

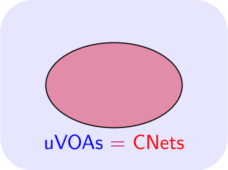
Every conformal net comes from a unitary VOA.



The situation before



The situation now



Conjecturally

The VOA $V \subset H$ lives on the space of *finite-energy vectors* for L_0.

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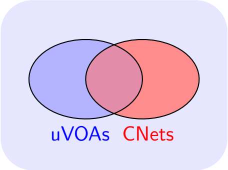
Every conformal net comes from a unitary VOA.



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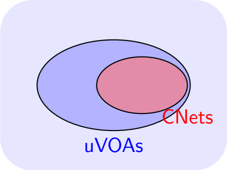

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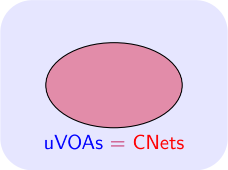

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We also show that:

- every representation of a CN comes from a VOA module.
- a number of constructions of CNs from VOAs are equivalent
- there are many conditions equivalent to a VOA producing a CN

Interlude: the semigroup of annuli

- The Virasoro algebra L_n is a central extension of the complexification of the Lie algebra of $\mathrm{Diff}(S^1)$.
- Goodman-Wallach '85: Every unitary representation of Virasoro exponentiates to a projective unitary representation of $\mathrm{Diff}(S^1)$.

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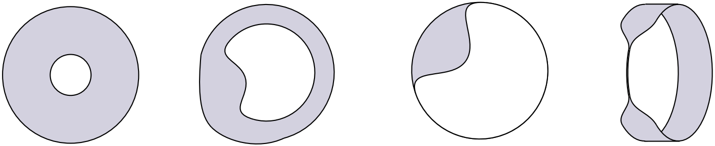
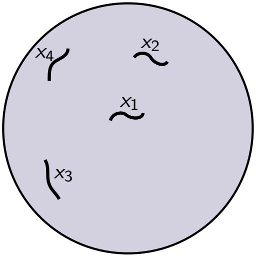
- Goodman-Wallach '85: Every unitary representation of Virasoro exponentiates to a projective unitary representation of $\mathrm{Diff}(S^1)$.
- There is no group which "complexifies" $\mathrm{Diff}(S^1)$, but there is a semigroup.
- The Segal-Neretin semigroup of annuli consists of certain complex 'manifolds' with boundary parametrized by S^1. We allow 'partially thin' annuli.
- Henriques-T: Every positive energy representation of Virasoro exponentiates to a holomorphic representation of this semigroup.



Worm insertions

- We want to construct a vector in H corresponding to vectors v_j inserted at points z_j in the unit disc (with local coordinates)
- We first construct vectors in H corresponding to algebra elements $x_j \in \mathcal{A}(I_j)$ inserted at intervals in the unit disc (parametrized by I_j)



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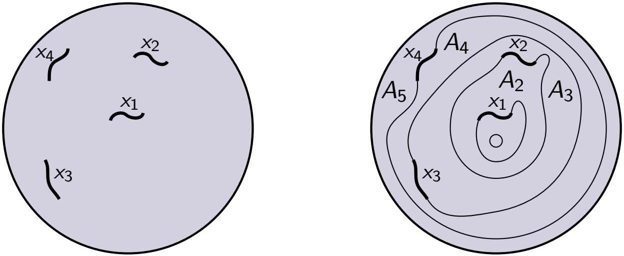
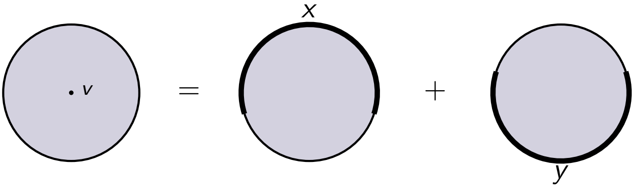


$$A_n x_{n-1} \cdots A_3 x_2 A_2 x_1 A_1 \Omega \in H$$

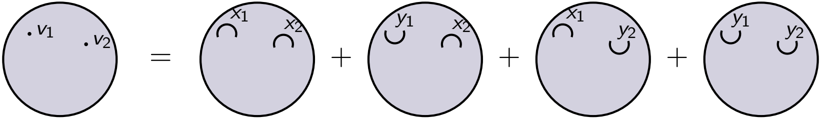
- Thin annuli are used to show that this is independent of choices.

Point insertions

- Lemma: If I and J cover S^1, then for any $v \in V$ there exist $x \in I$ and $y \in J$ such that $v = x\Omega + y\Omega$



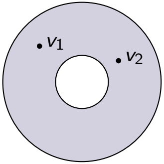
- The vector assigned to a configuration of n labelled points (v_j, z_j) is a sum of 2^n terms:

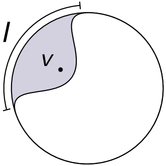


- This is again independent of choices.
- There is a lot to check, but these are the fields of a VOA.

Annuli with insertions

- Given a conformal net, we can assign a bounded operator $H \to H$ to an annulus with point insertions from V.
- This is first done for worms x_j, making lots of choices, and then point insertions are a sum of 2^n worm insertions.



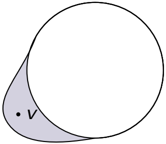


- If the annulus is "suported in I", then the corresponding operator lies in $\mathcal{A}(I)$.

- The algebras $\mathcal{A}(I)$ are *generated* by these point insertions.

Annuli with insertions - VOA

- Starting with a unitary VOA, there is a different way to define an operator corresponding to an annulus with point insertions:



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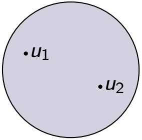

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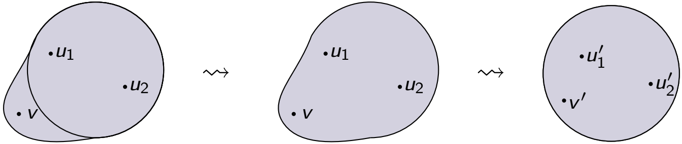

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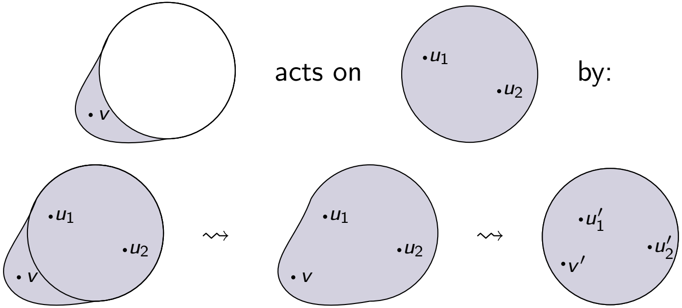

- This is only defined on linear combinations of point insertions.

- The VOA comes from a conformal net if and only if this extends to a continuous operator $H \to H$.

- In this case, the two different actions of annuli with point insertions agree.

Thank you!