Conformal nets are geometric!

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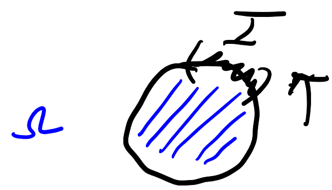
(joint work with André Henriques)

A <u>conformal net</u> consists of:

- a Hilbert space $\mathcal{H}$, a vector $\Omega \in \mathcal{H}$
- a projective representation $U : \mathrm{Diff}_+(S^1) \curvearrowright \mathcal{H}$
- von Neumann algebras $\mathcal{A}(I) \subset B(\mathcal{H})$ for all intervals $I \subset S^1$



such that

- $I \subseteq J \Rightarrow \mathcal{A}(I) \subseteq \mathcal{A}(J)$
- $U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma(I))$
- Ω is fixed by $\mathrm{M\ddot{o}b} = \mathrm{Aut}(\mathbb{D})$, cyclic for $\bigvee \mathcal{A}(I)$
- $I \cap J = \emptyset \Rightarrow \mathcal{A}(I) \subseteq \mathcal{A}(J)'$
- if $\gamma|_I = id$, then $U(\gamma) \in \mathcal{A}(I)'$
- the generator L_0 of rotation is positive

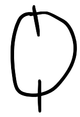
Consequences

• $\mathcal{H} = \bigoplus^{\ell^2}_{n\in\mathbb{Z}_+} V(n)$ ← $\mathrm{Ker}(L_0 - n)$

→ $V := \bigoplus^{alg}_{n\in\mathbb{Z}_+} V(n)$ finite energy

We assume $\dim V(n) < \infty$; conjectured to always hold

• $A(I)$ is the hyperfinite
→ III_1 factor

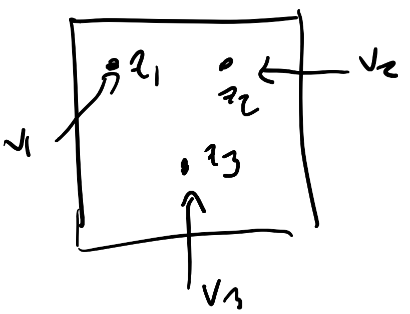


• $A(I') = A(I)'$
↗

A representation of a conformal net is a family of representations $\pi_I : A(I) \to B(\mathcal{H}_\pi)$, compatible with inclusions $I \subset J$. $\left[\pi_J|_{A(I)} = \pi_I\right]$

subfactor: $\pi_I(A(I)) \subseteq \pi_{I'}(A(I'))'$

$\mathcal{H}$, V, $\mathcal{U}$, etc as above



$\bullet\ \infty$
u

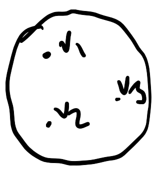
local coordinate

$z_1 \mapsto 0$

Correlation function $\mu((v_1, z_1), (v_2, z_2), \ldots, (u, \infty))$

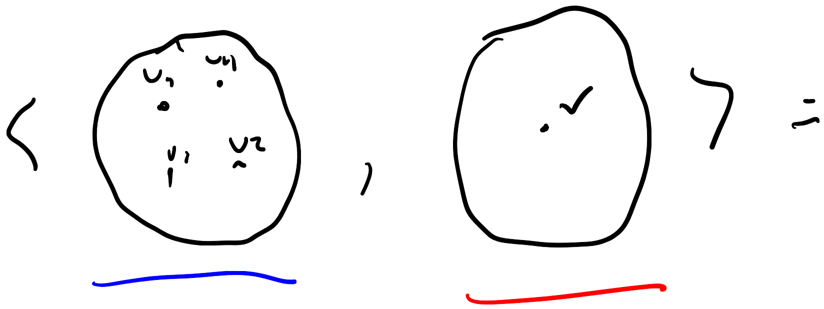
Holomorphic in z_i. Primary object of physical study.

Mathematically, these are axiomatized by a vertex operator algebra. "unitary" VOA ⇝ there's an inner product
$v_i \in V$



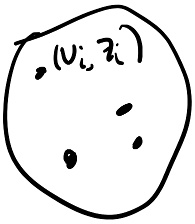
⇝ vector in $\mathcal{H}$.

$\longleftrightarrow v \in V$

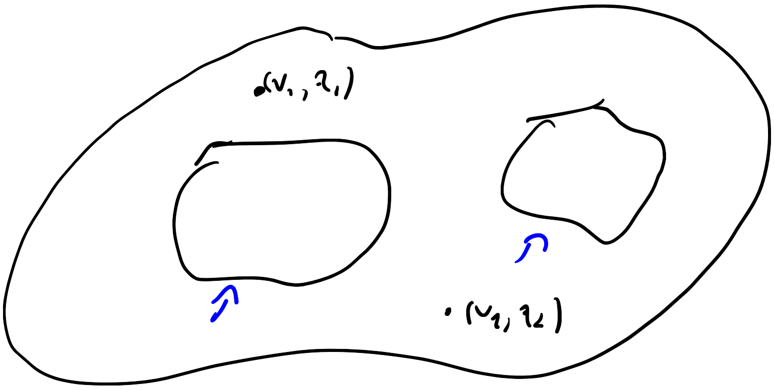


correlation function assigns a number

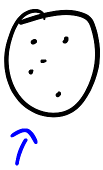
Unitary VOAs give a holomorphic function



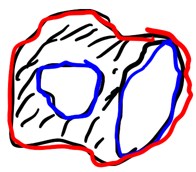
$\longmapsto$ vector in $\mathcal{H}$



$\Big\{$ surface w/ marked points, ∂ parametrized by S^1

$\mathcal{D} = \text{span}\Big\{$  $\Big|\ v_i \in V,\ z_i \in \mathbb{D} \Big\}$

$Y_\ell : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$

We also get maps when Σ is this 

Def A unitary VOA is *integrable* if the operators $\xrightarrow[v_i]{z_i}$: $\mathcal{D} \to \mathcal{D}$ are bounded.

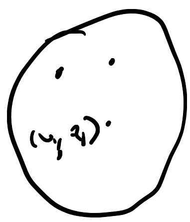
Theorem [Henriques - T]

There is a bijection between conformal nets and integrable unitary VOAs, given by

$$A(I) = \left\{ \;\_{I'} \;\middle|\; \begin{array}{l} \operatorname{supp}(\Sigma) \subseteq I \\ v_i \in V \\ z_i \in \Sigma \end{array} \right\}''$$

Σ

How to construct

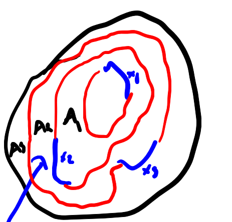


$\longmapsto$ Vector in $\mathcal{H}$

"disks w/ point insertions"

disk $\mathbb{D}$
with labeled points
(v_i, z_i)

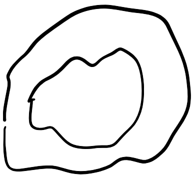
First, "disks w/ worm insertions" $\rightsquigarrow$ Vector in $\mathcal{H}$

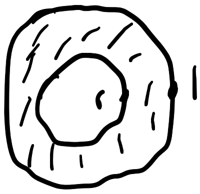


$(I, \; x \in A(I))$

Bartels - Douglas - Henriques:
abstract vNA associated with abstract interval I

Segal - Neretin semigroup of annuli
Every positive energy representation of $\mathrm{Diff}(S^1)$ extends to a rep of the semigroup of annuli

A_n : {  }   Complex annuli with parametrized boundary


w/ standard parametrization $\mapsto r^{L_0}$

$(r<1)$

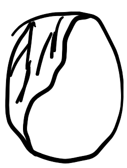

rotation $= e^{i\theta L_0}$

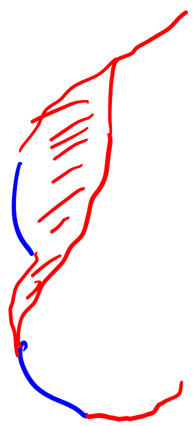
analytically continue to $t = i\theta$,

$\mapsto A_3 x_3 A_2 x_2 A_1 x_1 \Omega \in \mathcal{H}$

Major step: Show this is independent of choices, with order being the biggest one

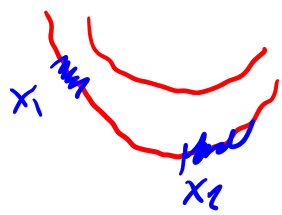
Theorem [Henriques–T] Every PER of $\mathrm{Diff}_+(S^1)$ extends to a representation of the semigroup of partially thin annuli



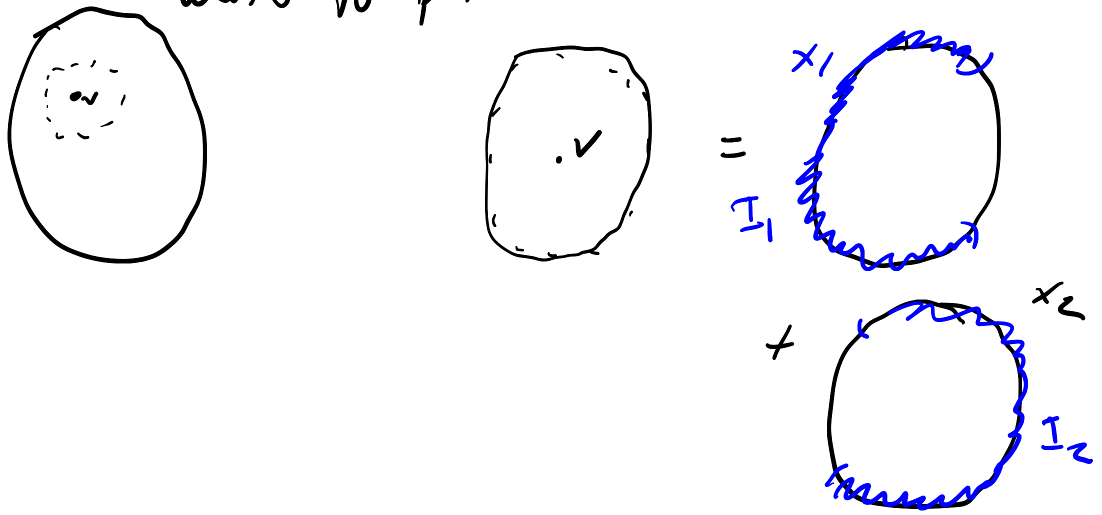


$$\frac{d}{dt} A(t) = X(t) A(t)$$

$=$

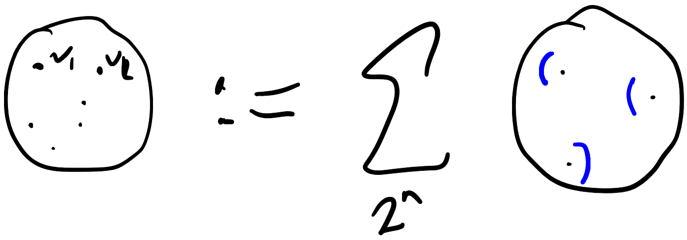


Vecs to points



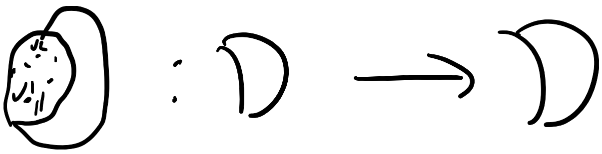
$\forall\ v \in V,\ \exists\ x_1 \in \mathcal{A}(I_1)$ and $x_2 \in \mathcal{A}(I_2)$ s.t.

$v = x_1\Omega + x_2\Omega$



Have to show this is well-defined.

This gives vectors 

and 

is bounded.

	CN	UVOAs
points in disks	vec in H_0	vec in H_0
points in annul	bdd $H \to H$	$\boxed{\text{map } D \to D}$

bdd $\iff$ integrable